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GAṆITA BHĀRATĪ
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THE GAṆITA KAUMUDĪ OF NĀRĀYAṆA PAṆḌITA

(ENGLISH TRANSLATION WITH NOTES)

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TRANSLATOR'S INTRODUCTION

Nārāyaṇa Paṇḍita lived in the 14th Century of the Christian era. He wrote his *Gaṇita Kaumudī* in 1356 A.D. He has given a very brief account of himself towards the end of *Gaṇita Kaumudī.*[1]

Gaṇita Kaumudī was translated by me while preparing for my Ph.D. Thesis "A Critical Study of the Contributions of Nārāyaṇa Paṇḍita to Hindu mathematics". under the guidance of Dr. K.S. Shukla (Lucknow Univ.). The thesis was presented to Bihar University Muzaffarpur, in 1977. The translation was revised while preparing for my D.Sc. Thesis "Certain Mathematical Processes as found in Hindu Mathematics" under the guidance of Late. Dr. S.R. Sinha (Allahabad University). This thesis was submitted to Allahabad University in 1985.

Nārāyaṇa has divided his *Gaṇita Kaumudī* in 14 chapters. First chapter of it deals with names of notational places, definition of weights and measures and logistics. Second chapter treats partnership, purchase and sale, interest and other miscellaneous topics. Third chapter contains arithmetic and geometric progressions. 4th chapter deals with plane figures, specially triangles and quadrilaterals. 5th, 6th, 7th and 8th chapters contain rules pertaining to excavations, piles of bricks and sawn pieces of timber, heaps of grain and shadows. The treatment of the topics in the above chapters is similar to that of Bhāskara II in his *Līlāvatī.*[2] Indeterminate equations of 1st and 2nd degrees are dealt in 9th and 10th chapters and here again the treatment is similar to Bhāskara's *Bījagaṇita*[3]. 11th chapter of *Gaṇita Kaumudī* gives rules for finding divisors of a number and is an extension of the work of Śripati on the same topic[4]. 12th chapter treats the topic of unit fractions and is similar to the treatment given by Mahāvīrācārya in his *Gaṇita Sāra Saṅgraha*[5]. 13th chapter contains rules on the topic of combinatorics (permutations, combinations, partitions, binomial and multinomial coefficient etc.), some series and, some other related topics. Lastly, 14th chapter of *Gaṇita Kaumudī* deals with magic squares. Magic square has not been dealt with by any other earlier Hindu mathematician.

In this work, the subject matter is first introduced before giving a rule (=R). After that the rule of the text is given in the form of translation within double inverted commas. After translation, each rule is interpreted in modern symbols and notations. Next, the rule is explained, if necessary. Variation of the rule given earlier by Hindu mathematicians together with matters of historical and other importance regarding the rule is given in the end. Similar rules given by earlier Hindu mathematicians are indicated in the footnotes collected at the end of each chapter.

The following abbreviations are used

Ā = *Āryabhaṭīya* of Āryabhaṭa I, edited and translated by K.S. Shukla and K.V. Sarma, INSA, New Delhi, 1976.

BSS = *Brāhma-sphuṭa-siddhānta* of Brahmagupta, edited with S. Dvivedi's Commentary, *Vāsanā,* and Hindi translation by R.S.Sharma *et al*, 4 Vols., New Delhi, 1966.

Ex. = Example (*udāharaṇa* or *uddeśaka*).

GB = *Gaṇita Bhāratī.*

GK = Ganita Kaumudī of Nārāyaṇa Paṇḍita, ed. by Padmakara dvivedi; Benares, Part I, 1935, Part II, 1942.

GSS = *Gaṇita-sāra-sangraha* of Mahāvira, ed. with Hindi translation by L.C. Jain,Sholapur, 1963.

GT = *Gaṇita-tilaka* of Śrīpati transl. by K.N. Sinha, *GB,* 4 (1982), 112-133.

HHM =*History of Hindu Mathematics* by B.B. Datta and A.N. Singh, Lahore, Part I, 1935, Part, 1938.

L = *Līlāvatī* of Bhāskara II, transl. by H.T. Colebrooke, Kitab, Mahal, Allahabad, 1967.

MS = *Mahāsiddānta* of Āryabhaṭa II, ed. by S. Dvivedi, Benares, 1910.

PG = *Pāṭī-gaṇita* of Śrīdhara ed. with an ancient com. and transl. by K.S. Shukla, Lucknow, 1959.

R = Rule (Sūtra) esp. from the *GK.*

SŚ = *Siddhānta - śekhara* of Śrīpati, ed. by Babuaji Misra, Part II, Calcutta, 1947.

Tri = *Triśatikā* of Śrīdhara, ed. by S. Dvivedi, Benares, 1899.

vs(s).= verse (s).

GAṆITA KAUMUDĪ

CHAPTER I

Nārāyaṇa begins the first chapter of his *Gaṇita Kaumudī* with a prayer to God (vs.1). Then he gives the names of notational places (vss. 2-3). After that he gives tables of money measures, weights, linear measures, measures of area, solid and capacity and time measures (vss. 4,5-6a, 6b,-7a, 7b-8a, 8b-10a, 10b-11a and 11b-12a, in order. Rules of calculation for addition and subtraction, multiplication, division, squaring, finding the square-root, cubing and finding the cube-root are given after that (vss, 12b, 13-15, 16, 17-18, 19-20, 21-23 and 24-25, in order). Next he deals with fractions. Here, he first gives rules for reduction of fractions and, simplifying different classes of fractions (vss. 26-27) and then, rules for addition, subtraction, and multiplication etc. of fractions (vss. 28-29). After that he

gives rule for operations, involving cipher (vs. 30), Rules of concurrence find place after that (vss. 31-37a). Next, he gives rules for the solution of linear equations, quadratic equations and double equations of second and higher degrees (vss, 37b-38, 39-44 and 45-58, in order). Next, he deals with rule of inversion, rule of three and inverse rule of three (vss. 59, 60 and 61, in order). Finally rule of five etc., rule of barter of commodities and rule for sale of living beings are given (vss. 62, 63a and 63b, respectively).

R.1. "Having bowed to the Almighty, (I) describe this Gaṇita Kaumudī for removing the darkness (of ignorance) and increasing the knowledge of spotless mathematics (which is like) a sea and (which is the life of) many people (just as *cakavā*) (is the life of) cakora."

R.2-3. "The names of notational places increasing successively in the ratio of ten are *eka* (unit), *daśa* (ten), *śata* (hundred), after that *sahasra* (thousand), *ayuta* (ten thousands), *lakṣa* (hundred thousands), *prayuta* (million), next koṭi (ten millions), *arbuda* (hundred millions); next *nikharva* (hundred thousand millions); after that *mahābja* (billion), śaṅku (ten billions), *pārāvāra (hundred billions), antya* (thousand billions), *madhya* (ten thousand billions) and *parārdha* (hundred thousand billions)."[6]

Counting from right to left, the units' place is called *eka*, the tens' place is called *daśa*, the hundreds place is called *śata* and so on. Terms upto *ayuta*, used in the above sense, occur frequently in *Ṛg-Veda*. the following list occurs in *Yajur-Veda* and is associated with the name of Medhātithi.[7]

Eka, daśa, śata, sahasra, ayuta, niyuta, prayuta, arbuda, nyarbuda, samudra, madhya, antya and *parādha* ($= 10^{12}$).

The list with slight variations is also found in other vedic works such as *Kāṭhaka Saṁhitā, Maitrāyaṇī Saṁhitā, Pañcaviṁśa Brāhmaṇa, Jaiminīya-upaniṣad-Brāhmaṇa* and *Śāṅkhāyaṇa Śraut Sūtra. Śatapatha Brāhmaṇa and Taittirīya Saṁhitā* give the list without any change.[8]

TABLE OF MONEY MEASURES

R.4. "20 *kapārdikās* make a *kākiṇi*, 4 *kākiṇis*, a *paṇa*; 12 *paṇas*, a *dramma* and 36 *drammas, a niṣka."*

Śrīdhara, Śrīpati and Bhāskara II replace *Kapārdika* by *varāṭaka*.[9] According to Śrīdhara, 16 *paṇas* make a *puraṇa* and according to Śrīpati 16 paṇas make a dramma. Their lists end here. According to Bhaskara II, 16 *paṇas* make *dramma* and 16 *drammas*, a *niṣka*.

TABLE OF WEIGHTS

R-6(a). "1/100 of a *tulā* is called *pala*. 1/4 of a *pala is named a* karṣa. That is also called a *suvarṇa*. 1/16 of a karṣa is a *māṣa* and 1/5 of a *māṣa*, a guñjā. 3 *guñjās* make a valla (and) 16 vallas, a gadyāṇa."

Thus the measures are,	5	*guñjās*.	=	1 *māṣa*
	16	*maṣas*	=	1 *karṣa* (*or suvarṇa*)
	4	*karṣa*	=	1 *pala*
	100	*palas*	=	1 *tulā*.

Also,	3 guñjās	=	1 *valla*
	16 *vallas*	=	*1 gadyāṇa*

Śrīdhara and Bhāskara II have not described tulā[10]. Likewise, barley corn, *dharaṇa* and *dhaṭaka* described by Bhāskara II have not been described by Nārāyaṇa.

TABLE OF LINEAR MEASURES

R.6 (b)-7(a). "24 *angulas* make a *hasta* (i.e., cubit) and 10 *karas* (or *hastas*), a *daṇḍa* (i.e., staff). 800 *daṇḍas* is a krośa and 4 krośas, a *yojana.*"

Daṇḍa of Mahāvīra, Śrīdhara, Śrīpati and Bhāskara II is smaller than that of Nārāyaṇa, the former being equal to 4 cubits[11]. Their other measures are the same as that of Nārāyaṇa. *Pāda* and *vitasti* described by Mahāvīra; *yava* and rajju described by Śrīpati and *yava* and *vaṁsa* described by Bhāskara II have note been described by Nārāyaṇa.[12]

TABLE OF MEASURES OF AREA AND VOLUME

R.7(b)-8(a),"Area of square with a side equal to 20 *daṇḍas* is a karṇaka. This is (also called a) *nivartana.* Area of (a sqaure) with a side equal to 400 *daṇḍas* is an *koṣṭha.*"

Koṣṭha has not been described by Bhāskara II[13].

R.8(b)-10(a). A solid having 12edges and length, breadth and height each equal to one cubit, say the mathematicians, determines a *ghanahasta.* A solid of length, breadth and height equal to 24, 16 and 16 *aṅgulas,* is the measure of a *dṛṣatkara*. A *ghanahasta* comprises of 2¼ *dṛṣatkaras.*"

Thus, 13824 cubic *aṅgulas* = *1* ghanahasta, and 6144 cubic *aṁgulas* = 1 *dṛṣatkara*

TABLE OF MEASURES OF CAPACITY

R.10 (b)-11(a). "20 *kuḍavas* make a *khārī* and 1/16 of a *kuḍava* is known as pādikā. 216 cubic *aṅgulas* is the measure of a *p*ādikā."

Thus, the measures are,	216	cubic *aṅgulas*	=	1 *pādikā*
	16	*pādikās*	=	1 *kuḍava*
	20	*kuḍavas*	=	1 *khārī*

Khārī used in Magadha as stated by Śrīdhara and Bhāskara II is equivalent to a cubic cubit. From definitions given above Nārāyaṇa's khārī is equivalent to 5 cubic cubits[14]. The term *kuḍava* (along with droṇa and *āḍhaka*) are also mentioned in *Ved*āṅga-Jyotiṣa[15].

TABLE OF TIME-MEASURES

R.11(b)-12(a). "60 *ghaṭik*ās is a nychthemeron (a day and a night); 30 of them make a month, 12 months, a year. These are the definitions stated in mathematics."

Minute divisions of time as stated below were prevalent in India, from very early times. Thus in *Śatapatha Brāhamaṇa*, we have,[16]

15	*praṇas*	=	1 *idanī*
15	*idanis*	=	1 *itarhi*
15	*itarhis*	=	1 *kṣipra*
15	*kṣipras*	=	1 *muhūrta*
30	*muhūrtas*	=	1 *divasa*

EIGHT OPERATIONS WITH WHOLE NUMBERS

Addition and Subtraction

R.12 (b). "(Digits of) the same denomination should be added and subtaracted in (numbers) of the same class."

Numbers of different classes must be converted into those of the same class before addition or subtraction. Also, in numbers of the same class, digits of the same denomination are to be added or subtracted. Thus, digits at the units' place are to be added (or subtracted) at a time and the sum (or the difference) gives the digit at the unit's place in the result and similarly for the digits at the tens' place and so on.

Two methods for addition and also for subtraction have been given by Śrīpati and Bhāskara II, both.[17] The methods have been explained by Datta and Singh.[18] Śrīdhara has described the methods with the help of series.[19]

Ex.1 "Tell the sum of 8, 4, 5, 7, 15, 324 and 26 *drammas* added to 2 niṣkas and the remainder after their subtraction from one thousand (*drammas*)."

Multiplication

R.13-15. "Keep the multiplier below the multiplicand as in the junction of doors, Starting from the last place, multiply the multiplicand by (the digits of) the multiplier, severally. (proceed) by removing (the digit) after multiplication and then moving (to the next digit). Alternatively, multiply the parts of the multiplier (obtained by the method of) *rūpabibhāga* by the multiplicand. The sum of the products is the result. Alternatively, multiply the parts (of the multiplicand or the multiplier, in multiplication by the method *sthānavibhāga*) according to places, by (the multiplier or the multiplicand, as the case may be). The sum of the products, according to (their) places, is also the result. Alternatively, (a number) exactly divisible by a divisor is the result of multiplication of that (divisor) and the quotient. There is no difference between the multiplicand and the multiplier and the multiplier multiplied by the multiplicand is the same (as the multiplicand multiplied by the multiplier)."

Nārāyaṇa has not given details of the processes. Datta and Singh have explained the first process and its variation as used by the Hindus, in detail.[20] Arabs learnt the decimal arithmetic from the Hindus and the *kapāṭa-sandhi* method of multiplication was transmitted to them from the Hindus.[21] The method occurs in the works of Al-Khwārizmī (825 A.D.), Al-nasavī (1025 A.D.), Al-Haṣṣār (1175 A.D.) Al-Kalasādī (1475 A.D.) and many others.[22]

In the second method, the multiplier is broken into two or more parts whose sum is equal to it. The multiplicand is then multiplied severally by these and the sum of the products gives the result. The method was transmitted to the Arabs and Italians from the Hindus.[23] Third method has been explained by Datta and Singh.[24] Fourth method was also transmitted to the Arabs and Italians from the Hindus.[25]

Some of these methods were given by Brahmagupta[26] and Āryabhaṭa II. First, second and third methods were also given earlier by Mahāvīra, Śrīdhara, Śrīpati and Bhāskara II.[28]

Ex. 2 "(What is the product of) 275 multiplied by 18? What are they (i.e., the products) obtained from parts produced by (the method of) rūpabibhāga, sthāna-bibhāga and by, division without remainder?"

Division

R. 16 "That number which (when) multiplied by the divisor balances the last digit of the dividend, is the partial quotient (and so on). Alternatively, (first) abridge the divisor and the dividend (both) by the same number (and then) proceed to division."[29]

The process as used by the Hindus has been explained by Datta and Singh.[30] It seems that the method was invented in India about the 4th century A.D. or even earlier, The method was transmitted to the Arabs where it occurs in the works of well known writers such as Al-Khwārizmī (c. 825), Al- Nasavī (c. 1025) and others.[31]. From there the method was transmitted to Europe.[32]

Squaring

R.17-18. "The product of two equal numbers is the square of the number.[33] Place the square of the last (digit) over itself and then the products of twice the last (digit) and the others (i.e., the rest) over themselves, respectively. Next moving the number obtained by leaving the last digit, repeat the process.[34]

Alternatively, the product of the sum and the difference of the number and an assumed number, added to the square of the assumed number gives the square (of the number).[35] The sum of as many terms of the series of which 1 is the first term and 2, the common difference, (is the square of the given number).[36] The square of the difference of two numbers added to four times the product of the number (is the square of their sum)."

Here, Nārāyaṇa gives five methods for finding the square of a number. According to the first, $n^2 = n \times n$. Second method has been described by Datta and Singh in detail.[37] Other methods are based on the formulae,

(a) $n^2 = (n-a)(n+a) + a^2$

(b) $n^2 = 1+3+5+$................upto n terms

(c) $(a+b)^2 = (a-b)^2 + 4ab$.

Ex. 3. "Friend, if you know, tell me severally the squares of 1 etc. upto 9, 17 and 10225."

SQUARE-ROOT

R.19-20. ("Mark the places) odd (or) even (as the case may be). Subtract the (greatest) square number from the last odd place. (Set down double its root underneath the next place in a line). Divide (the remainder upto) the even place by twice the root and (then) subtract the square of the quotient from the next odd place. Set down twice the quotient in the line (of double the square-root). Move that (i.e., the resulting quantity in the line of double the square-root, one place forward). In this way divide again and again (till all the places are exhausted). Half of the number in the line of the (double-) root becomes the square-root."[38]

Hindu method of extracting the square-root of a number has been given by Shukla in detail.[39] This method of extracting the square-root seems to have been communicated to the Arabs about the middle of the 8th century for it occurs in the same from in Arabic works on mathematics.[40] In Europe it occurs in similar forms after 14th century.[41]

CUBING

R.21-23. "The (continued) product of three equal numbers is the cube (of that number).[42] Set down the cube of the last. Multiply the squares of the last and that preceding the last by the first (and) the last (respectively and then multiply both the products) by 3. (Set down these) and the cube of the first (in order, such that) there is a difference of 1 place (between one result and the other and) add. (The sum) is the cube (of the combined number formed) by the last and the first. (If there be more than 2 digits in the given number, treat the combination of the last and the first as the last and proceed further and so on).[43] Form the series whose first term and common difference are 1 (each). Cube of the preceding (term) added to 'thrice the last (term) multiplied by the preceding (term)' and, to 1 is the cube of the last."[44]

"Or, thrice the (given) number multiplied by the 2 parts, (the product) added to the cubes of the parts, is the cube of the sum (of the parts). [45] The square of the cube of the (square-) root of a number is equal to the cube (of that number)"[46]

Here, Nārāyaṇa gives 5 methods for finding the cube of a number. According to the first, $n^3 = n \times n \times n$. Second method has been described by Shukla in detail.[47] Other methods are based on the formulas,

(a) $n^3 = (n-1)^3 + 3n(n-1) + 1$

(b) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(c) $n^3 = \left[\left(\sqrt{n}\right)^3\right]^2$

Ex.4 "Tell quickly the cubes of 1 upto 9 and also, of 18,36,361 and 507."

CUBE-ROOT

R.24-25. "(Divide the digits beginning with unit's place into periods of) one

'cube' place and two 'non-cube' places. Subtracting the (greatest possible) cube from the (last) 'cube' place and keeping the root at another place divide the (next non-cube place) by thrice the square of the root and keep that (i.e., the quotient) separately (in the line of the root next to the root obtained) prior to that. Subtract 'thrice the square of that (i.e., the quotient) multiplied by the last (i.e., the cube-root)' from the next non-cube place and (then subtract) the cube of the quotient from the next cube place (and so on)."[48]

The process has been described by Shukla in detail.[49]

FRACTIONS

Fraction have been used in India from very early times. In *Ṛgveda*, the fraction *ardha* (i.e.,½) and tri-pāda (i.e., ¾) occur.[50] The fractions kalā (i.e., 1/16), *kuṣṭha* (i.e.,1/12), śapha (i.e., 1/8) and *pāda* (i.e., ¼) are mentioned in *Maitrāyaṇī Saṁhitā*. In *Śulba-sūtra*, the earlierst known mathematical work, fractions have been used in the statement and solution of problem.[51] Hindu mathematicians used to divide fractions into different classes, probably according to the form in which these were used to be written and their nature. There is no uniformity about it.[52]. Nārāyaṇa divides them into six classes.

RULE FOR *BHĀGAJĀTI ETC.*

R.26 (a) - 26 (b) (i). "To reduce fractions to a common denominator, divide (each of) the denominators by the common factor (if any and then) multiply the numerator and the denominator (of each of the fractions) by the denominator of the others.[53] In case of an integer, unity is the denominator."[54]

Ex.5, "1/2,1/4,1/6 (and) 1/12 are reduced to a common denominator. Tell these. 2/3, 3/5 (and) 5 (are also reduced to a common denominator). Tell. these."

REDUCING FRACTION OF *PRABHĀGA* CLASS

R.26b(ii). "In (fractions of) *prabhāga* (class) multiply the numerators and the denominators."[55]

Ex. 6. "O learned, if you are well versed in (fractions of) prabhāga (class), tell (the amount) given by a greedy man to a beggar on repeated requests with respect, (if he) gave,

$$\frac{1}{16} \text{ of } \frac{1}{9} \text{ of } \frac{1}{8} \text{ of } \frac{1}{2} \text{ of } \frac{1}{6} \text{ of } \frac{2}{3} \text{ of } 1 \textit{ niṣk.''}$$

REDUCING FRACTION OF *BHĀGĀNUBANDHA* AND BHĀGĀPAVĀHA CLASSES

R. 27(a) "Having multiplied the integral part by the denominator, add to the product obtained by multiplication (or) subtract from (it, as the case may be, the numerator of the fraction)."[56]

Ex.7(a). "If you know simplification of fractions, (simplify and) tell quickly, 3+1/5, 6+3/7, 2-1/3 and 5-1/4."

R.27(b). "Multiply the initial (i.e., upper) denominator by the lower denominator. Add (or subtract) one's own (i.e., lower) numerator to (or from, as the case may be) one's own (i.e., lower denominator. Multiply (the result) by the initial (i.e., upper) numerator."[57]

The Hindus wrote fractions without the dividing line. However, when several fractions occurred in the same problem, they were in general separated from each other by vertical and horizontal lines.[58] Thus, $\frac{a}{b}+\frac{c}{d}$ of $\frac{a}{b}$ used to be written as $\begin{matrix} a \\ \frac{b}{c} \\ d \end{matrix}$ or $\begin{matrix} a \\ b \\ c \\ d \end{matrix}$

(hence the names, upper numerator, and lower numerator etc.) Dr. Shukla has dealt with the Hindu process of reducing such fractions in detail. [59]

Ex. 8. "Tell, O friends, what is $\frac{1}{4}+\frac{1}{3}$ of $\frac{1}{4}+\frac{1}{5}$ of $\left(\frac{1}{4}+\frac{1}{3}\text{ of }\frac{1}{4}\right)$ and $\frac{1}{7}+\frac{1}{4}$ of $\frac{1}{7}+\frac{1}{6}$ of $\left(\frac{1}{7}+\frac{1}{4}\text{ of }\frac{1}{7}\right)$ made homogeneous, in order.

Tell, if you know, (what is)

$$\frac{2}{3}-\frac{1}{2}\text{ of }\frac{2}{3}-\frac{1}{4}\text{ of }\left(\frac{2}{3}-\frac{1}{2}\text{ of }\frac{2}{3}\right),\text{ and}$$

$$\frac{5}{6}-\frac{1}{6}\text{ of }\frac{6}{5}-\frac{7}{9}\text{ of }\left(\frac{6}{5}-\frac{1}{6}\text{ of }\frac{6}{5}\right)."$$

ADDITION AND SUBTRACTION OF FRACTIONS

R.28(a). "Reduce the fractions to a common denominator by the method stated earlier (and then) add or subtract (as the case may be)."[60]

Ex. 9. "O learned, tell quickly the sum of $\frac{1}{15}, \frac{1}{6}, \frac{1}{10}$ (and) $\frac{1}{3}$. If you have laboured in fractions, tell, what is the remainder (when) these are subtracted from 1."

MULTIPLICATION OF FRACTIONS

R.28(b). "In multiplication of fraction 'the product of the numerators' divided by 'the product of the denominators' is the result."[61]

Ex. 10. "O friend, what is (the product of) $5\frac{1}{3}$ multiplied by $4\frac{1}{8}$ and (also of) $(2 - \frac{1}{5})$ multiplied by $(1+\frac{2}{3})$."

DIVISION OF FRACTIONS

R.29(a). "Having interchanged the denominator and the numerator of the divisor, apply the process (of multiplication) stated earlier."[62]

Squaring etc of fractions

R.29(b). "Having squared and cubed (as the case may be), the numerator and the denominator of the root, (divide the former by the latter) and in (square-root and cube)root, proceed (by the method) stated earlier."[63]

Ex.11. "Tell the square of $5\frac{1}{3}$ and, the square - root from (the resulting) square. Friend, if you know (operations for) fractions, (tell) the cube (of that and) the cube-root from that (cube)."

Operations involving zero

R.30. "A number remains unchanged (when) cipher is added to (or) subtracted from (it) and that (i.e., the number) multiplied by cipher becomes cipher. In multiplication etc. of cipher by a number, (the result) happens to be cipher. (when a number) is added to cipher (the result) is that which is added."[64]

Ex.12. "What (is the result when) cipher is added to (or) subtracted from 100 and, that (i.e., 100 is) multiplied by cipher? Also, tell me quickly, what happens (when) cipher is multiplied by cipher and cipher is divided by cipher? What is the square, the square-root, the cube and the cube-root of cipher ? O learned, if you are well versed in operations with cipher (tell, what is obtained when) 8 is added to cipher."

Here, it may be pointed out that $\frac{0}{0}$ is indeterminate. Nārāyaṇa comments, "Here, in *pāṭīgaṇita* (arithmetic) division by cipher is not recognised and therefore, it is not mentioned here. As it is of use in *bījagaṇita* (algebra) so I have narrated division by cipher in my *Bījagaṇita*."

He has treated the topic 'operations for cipher' in an exhaustive manner in his work on algebra, the *Bījagaṇitāvataṁsa*.[65] Prior to him Brahmagupta[66] and Bhāskara II[67] have treated the topic in algebra.

Concurrence (Saṅkramaṇa)

R.31. "Add and subtract the sum and the difference (of two numbers) kept at two places. Halves of these two (results) are the numbers. Due to this reason, this (process) is called *Saṅkramaṇa*, *Saṅkrama* and *Saṅkrāma*."[68]

Let $x + y = a$ and $x - y = b$. Then, according to the rule,

$$x = (a + b)/2 \text{ and } y = (a - b)/2$$

Ex. 13. "The sum of two numbers is 63 (and) the difference is 9. O Learned, if you know the method of concurrence, tell those two numbers, quickly."

Concurrence in (case of) difference

R.32. "The difference of squares divided by the difference of two numbers happens to be the sum and the difference (of squares when) divided by the sum

happens to be the difference. From them (i.e., from the sum and the difference) the numbers are obtained by the method of concurrence."[69]

Let $x^2 - y^2 = a,\ x - y = b$ and $x + y = c$

According to the rule, $x + y = a/b$ and $x-y = a/c$ Now, x and y can be obtained separately by the method of concurrence when a and b, or a and c are given.

Ex. 14 (a). "The difference of squares of two numbers is 400. The difference between the numbers is 8 or (their) sum is 100. O friend, if you know. tell them."

R.33. "The square-root of the difference between 'twice the sum of squares (of two numbers' and) 'the square of (their) difference' is their sum."[70]

Let $x^2 + y^2 = a$ and $x - y = b$. Then, according to the rule, $x - y = \sqrt{(2a - b^2)}$. Now, x and y can be obtained separately, by rule 31.

Ex.14 (b). If you know mathematics, tell quickly the two numbers, the sum of (whose) squares is 100 (and) the measure of whose difference is 2."

R.34."Supposing the product of the squares (of two numbers) the product and the difference of (their) squares as the difference, (the squares of) the numbers are (obtained from them (i.e., from the product and the difference) by the method of concurrence. (Square-) roots of (the squares give) those numbers, separately."

Let $x^2 - y^2 = a$ and $x^2y^2 = b$. According to rule 35,

$$(x^2+y^2) = [(x^2 - y^2)^2 + 4x^2y^2]^{1/2} = c, \text{ say;}$$

Now from a and c, x^2 and y^2 can be obtained, (rule 31) and from them x and y can be obtained, separately.[71]

Ex.14 (c) "O child, if you know, tell those two numbers the difference of whose squares is 175 and whose product is 300."

R.35. "The square-root(of the sum) obtained by adding 'the square of the difference of two numbers' to 'four times (their) product' is (their) sum. The square-root (of the difference) obtained by subtracting, 'four times the product (of two numbers) from ' the square of (their) sum' is (their) difference."

Let $x + y = a$, $x - y = b$ and $xy = c$. According to the rule, $x + y = (b^2 + 4c)^{1/2}$, and also, $x - y = (a^2 - 4c)^{1/2}$. After obtaining $x + y$ or $x - y$, as the case may be, x and y can be obtained, separately by[72] rule 31.

"What is the sum (of two numbers) whose product is 60 (and whose) difference is 7. If you know *sankrama*, tell their difference by knowing their sum."

R.36 "The square of the difference of two numbers added to twice (their) product is (equal to) the sum of their squares. The square-root of 'twice (their) product added to that (i.e., to the sum of the squares) is equal to the sum of the squares) is equal to the sum (of the numbers)."

According to the rule, $x^2 + y^2 = (x - y)^2 + 2xy$ and $x + y = (x^2 + y^2 + 2xy)^{1/2}$

Now, if $x - y$ and xy are known, x and y can be obtained, separately, by rule 31.

Ex. 16. "O friend, if you know, tell the sum of the squares, the square of the sum and the sum of two numbers, whose difference is 5 and whose product is 300."

R.37(a). "The square-root of 'the square of the sum (of two numbers) subtracted from twice the sum of their squares' is (their) difference."[73]

According to the rule, $x - y = [2(x^2 + y^2) - (x + y)^2]^{1/2}$

Now, since $x + y$ and $x - y$ are known, x and y can be obtained separately by rule 31.

Ex. 17. "If you know *Saṅkramaṇa*, tell the two numbers quickly, whose sum is 14 (and) the sum of whose squares is 100."

JĀTISAMUDĀYA

R.37 (b)-38. "Supposing unity as the number to be investigated, addition, subtraction, multiplication, division, ratio (and proportion) etc. are done (on unity) according to the statement of the problem. The result derived from unity is the divisor of the observed number. (The quotient gives the desired number)."[74]

Ex.18. "The sum of 1/3, 1/4 and 1/12 of a number (along with itself or) 1/4 of (the number along with itself) together with 1/3 of (the sum of) 1/12 and 1/4 of (the number) with itself added to twice their difference is 60. After that the fractional parts are subtacted from the numbers. (The resulting numbers) are observed to be 12, 18 (and) 12, (in order). Again), in (the above) 6 (problems), the numbers along with their (square-) roots are observed to be 6 more (or) less their (square-) roots are observed to be 6 less (than that observed earlier). What are (the numbers) in (each of) 3 cases of 6 types (of problems)."

Here, in 18 (a), 1+1/3 + 1/4 + 1/12 = 20/12, observed number (ob. no.) = 60 and so, the number $\left(\frac{60\text{x}12}{20}\right) = 36$;

in 18 (b), $1+\frac{1}{4}+\frac{1}{3}(1+\frac{1}{4}) = \frac{20}{12}$, ob.no. = 60 and so, the number $\left(=\frac{60\text{x}12}{20}\right) = 36$;

in 18(c), $1+\frac{1}{12}+\frac{1}{4}+2\left(\frac{1}{4}-\frac{1}{12}\right) = \frac{20}{12}$; ob.no. = 60 in18(d), $1-\frac{1}{3}-\frac{1}{4}-\frac{1}{12}=\frac{1}{3}$; ob no = 12, so the number $\left(=\frac{12\text{x}3}{1}\right) = 36$;

in 18 (e), $1-\frac{1}{4}-\frac{1}{3}\left(1-\frac{1}{4}\right) = \frac{1}{2}$, ob. no. = 18 and so the number (=18x2) = 36 ;

in 8 (f), $1-\frac{1}{4}-\frac{1}{12}-2\left(\frac{1}{4}-\frac{1}{12}\right)$; ob. no. = 12

and so the number (= 12x3) = 36.

Nārāyaṇa calls problems 18(a) to 18(f) as of *dhanaṁśa, dhanasvaṁsa, viśleṣa, ṛṇa, bhagapavaha* and *ṛṇaviśleṣa* classes of problems, in order. His problems. 19 to 24,given later on also belong to the same classes, in order. These arithmetical problems depend upon the solution of linear equations.

Mahāvīra and Śrīdhara have divided such problem into two classes, the former, into *bhaga* and *śeṣa* classes and the latter, into *stambha* and *viśeṣa* classes.[75] Śrīpati and Bhāskara II have divided them into three classes, *dṛśya, śeṣa and viśleṣa* classes.[76]

Next part of the problem when the roots are added or the last part, when the roots are subtracted, depends upon the solution of quadratic equations. Methods for the solution of problems based on quadratic equations have been dealth with later on in rule 39 onwards. The solutions of 2 out of these 12 problems and 3 other problems similar to these have been given (in short) by Nārāyaṇa after examples 28, 29, 30, 32 and 33.

Ex.19. "At the time of producing fragrance a swarm of large black bees were in an enclosure. 1/4 and 1/12 parts of them went outside (and then came and) mixed (with them), quickly. (Thus they became 16 in all). O mathematician, tell me the number of bees in the swarm remaining previously."

Ex.20. "A flock of geese flying in the sky with its one-third along with their half and again with 6 were observed to be 30. O friend, if you have laboured (in calculation) tell, how many birds were there in the flock."

Ex.21. "A flock of cuckoos (is sitting) in (a field of) rice with new foliage and buds. 1/3 thereof joins them coming from a mango tree, 1/8 (of them) from *Kikkila,* thrice their difference from a flowered *Ashoka* tree. O learned friend, if they became 100, (tell) quickly, how many cuckoos were there in the earlier flock."

Ex.22. "Half, 1/4 and 1/6 of a group of beautiful ladies, bowing down under the weight of their stout (and) lofty breasts, left and a lady well versed in musical instruments happens to be there. O learned, tell how many of them were there in the group."

Ex.23. "A pond whose water had become tawny due to the pollen of lotus, (was) full of geese roaming with pleasure (and) , abode of wandering large black bees. Here, being afraid after seeing (the geese), 1/5 of the bees along with 1/4 of the rest fled away. 1/3 of the rest and 1/2 of the remaining were wandering in *Kutaja*. 6 of the large black bees were flying in the sky. O friend, how many of them were there."

Ex.24. "1/2 of a herd of elephants has gone to (a pond) full of lilies. 1/5 of the remaining half are absorbed in plantain trees. Half of the half of that (which) remains is busy in sawallowing *Śallaka* trees growing on the peak of high mountain. 5 elephants together with a she-elephant (are seen there). How many of them were there in the herd."

QUADRATIC EQUATION

Altar-construction of the Vedic Hindus involved the solution of quadratic equations.[77]

Geometrical solution of simple quadratic equation is found in early Jain canonical works.[78] Aryabhaṭa I gives the solutions of quadratic equations while dealing with arithmetic progression and in relation to an interest problem.[79] Brahmagupta has given two rules for the solution of a quadratic equation.[80]

EQUATION OF SVARṆAJĀTI

R.39-40(a). "*Pada* and *agra* divided by *rupottha* happens to be the (so called) difference and 'the (so called) product' (respectively). Calculate 'the (so called) sum' from them by the method stated earlier. The numbers are (obtained from them) by the method of concurrence. (In case) *pada* is negative, the greater number is to be taken). The square of that is the number."

The rule gives the solution of a quadratic equation, $ax \pm b\sqrt{x} = c$.

Here, a is rupottha, b, the *pada* and c, the dṛśya (visible quantity). According to the rule, b/a gives 'the (so called) difference' and c/a, 'the (so called) product'. Now 'the (so called) sum' is to be calculated by rule 35 and so, it is $\left(\sqrt{b^2+4ac}\right)/a$. Again, the root is to be obtained by rule 31 and therefore, it is $\left(-b+\sqrt{b^2+4ac}\right)/2a$. In case b is negative, the greater number is to be taken and, so, the root in that case is $\left(+b+\sqrt{b^2+4ac}\right)/2a$. The sign of $4ac$ has come as positive because c has been taken on the right side of the equation. .

Ex. 25. "A group of cranes along with 1/4, 1/2 and the square-root (of the group) arrive in a pond. Cranes numbering 32 are (now) moving among flowering lotus flowers. O friend, tell the strength of the earlier group."

Ex. 26. "On the battlefield occupied by Duryodhana, O brave warrior, a group of warriors with its half along with 1/5 of them and (also) twice the square-root (of the warriors in the group) came near Abhimanyu who is full of anger due to irresistible attack of the warriors. If the (total) warriors were 200, tell how many of them were there in the group."

Ex. 27. "A group of peacocks along with its half, its one-third and 3/4 of their difference (i.e., difference between it half and its one-third) together with 1½ times the square-root (of the group) standing on the top of a mountain, hearing the thunder of clouds surcharged with electricity, are dancing with joy. The peacocks are 300 in all. O friend, tell me (the number of peacocks) in the earlier group."

Ex. 28. "A half along with 1/4 (of a group) of intoxicated elephants, being afraid of ferocious roar of a lion, fled towards the ridge of a mountain (and) 1/5 (of the group) less 1/10 of its square - root went away towards Kaccha state, happily. One chief elephant with 5 of its spouses remained there. O friend, how many of the elephants were there in the group."

Ex. 29. "(Out of a group of) birds a half and 1/3 of (of those which) remains together with '1/4) (of those) remaining (after that) less 1/4 (of those) remaining (after that) less 1/6 of the square-root (of the group) are roaming in the sky with

curiosity and 10 (birds) are observed on the land. Tell, how many birds were there (in the group)."

Ex.30 "With unmeasurable strength, Bhīma destroyed 1/2 and 1/3 of (a group of) elephants of the king of Kalinga with his club and, their difference less 1/.2 of the square-root (of those in the group) by slap as a weapon. 1 (elephant) with its teeth resembling those of a boar along with 5 remained. Friend, how many of them (were there in the group)."

Ex. 31. "In the battle for the destruction of the garden in Laṁkā, guards present there added to 1/8 and 1/18 (of them) were torn as under by Hanumāna at the play-ground. (Out of these) 14 times the square-root died after being tied by a tree and 2 were beaten by hand. How many of them were there?"

Ex.32 "(A here of) elephants along with 1/4 (of the herd) together with 1/5 of them (i.e., 1/5 of the total) were roaming. 5 times the square-root (of the herd) moved away for fear of lion. An elephant along with his 3 spouses were observed on the bank of the pond. How many of them were there (in the herd)."

Ex.33, "A group of ladies along with 1/3 and 1/9 of them together with their difference (i.e., the difference between 1/3 and 1/9 of them) were engaged in the pleasure house. Thrice the square-root (of the group) were engaged in playing with water and 6 were with their lover. Tell, how many of them (were there in the group)."

Ex.34. "1/5, 1/2 and 1/10 of a heap of lotus flowers were offered to Śiva, Hari and Durgā (respectively) and square-root (of the heap) along with its half, at the feet of Ramā. 2 lotus flower were offered to Gaṇeśa, 2, to the Sun (and) 1 lotus flower, (at the feet of) the teacher. My dear child, if you know, tell quickly the total number of lotus flowers in the heap."

Ex. 35. "1/5 of a swarm of large black bees are on *Kutaja* flowers, 1/4 of the rest, on *Sarala* (tree), 1/2 of the rest on clove tree, 1/5 of the square-root, on the blossoming *Bakula* (tee and) 2 pairs, on (another) clove (tree). O friend, tell me, (how) many of the black bees (were there) in the swarm."

Ex. 36. "A number of ladies reside in the palace of a king. 1/3, 1/4 and 1/6 of them are for management. 2½ times the square-root are well versed in singing in melodious voice. 3 are expert in telling stories. 2 are attendants of the female compartment and 1 is the queen. If (you) are well versed in arithmetic, tell, how many of them (were there)."

EXAMPLE IN *ADṚŚYA JĀTI*

Ex.37. "The lotus eyed damsel beat her lover on the head by a blue lotus. ½ and 1/4 of the dry petals are on the lotus stalk and twice the square-root fell on the ground. How many of them were there?"

EXAMPLE ON *NIRAṀŚA JĀTI*

Ex.38. "3 times the square-root of a group of young girls are in a painter's studio and a square-root (of the group are preparing garlands. 5 (of them) are playing with the lover. (How many of them were there)"

EQUATION OF *ŚEṢAMŪLA JĀTI*

R.40(b). "Starting from the end by the method stated earlier, the process, (on being performed on) the remainders, gives the number."[81]

The rule is meant for solving equation of the type

$$(x-a\sqrt{x}) - b\sqrt{(x-a\sqrt{x})} - \sqrt{x-a\sqrt{x}-b\sqrt{(x-a\sqrt{x})}} + \ldots = c, \quad \ldots(1)$$

Let $$x-a\sqrt{x}=y \quad \ldots(2)$$

and $$y - b\sqrt{y} = z \quad \ldots(3)$$

then, from (1) $$z-\sqrt{z}=c \quad \ldots(4)$$

According to the rule, first equation (4) is solved and then with its help, equation (3) and then, equation (2) which gives the value of x.

Ex.39. "As the king went out for hunding, the hunters quickly laid the trap. There, 1/3 of the deers netted themselves. Thrice the square-root of the rest along with 4 were troubled by the ferocious lion. O learned, tell, (how many of them were there)."

Ex.40. "During an amorous tussle (the beloved's garland of pearls was broken. Twice the square-root less 1/4 (of the root) of the pearls were on the cover of the bed. The square-root of the rest along with 2/3 (of this root) were seized by the lover. The square-root of the rest fell down on the earth and 2 (pearls) were in the staring (of the garlnand). Tell, how many (pearls were there in the garland)."

R.41(a). "Multiply' *rūpottha'* by the first remainder (and) the (the product) to the last remainder. (After that solve the problem) by the method stated earlier."[82]

The rule is meant for solving equations of the type.

$$x-a-\frac{b}{f}(x-a)-\frac{c}{g}\left\{x-a-\frac{b}{f}(x-a)\right\}-\frac{e}{h}\left[x-a-\frac{b}{f}(x-a)-\frac{c}{g}\left\{x-a-\frac{b}{f}(x-a)\right\}\right]\ldots-j\sqrt{x}=d$$

i.e, $$(x-a)\left(1-\frac{b}{f}\right)\left(1-\frac{c}{g}\right)\left(1-\frac{e}{h}\right)\ldots-j\sqrt{x}=d.$$

This on simplification leads to

$$x\left(1-\frac{b}{f}\right)\left(1-\frac{c}{g}\right)\left(1-\frac{e}{h}\right)\ldots-j\sqrt{x}=a\left(1-\frac{b}{f}\right)\left(1-\frac{c}{g}\right)\left(1-\frac{e}{h}\right)\ldots+d.$$

Here 'a' is the first remainder and d the last remainder. After multiplying a by *rupottha* (i.e., the coefficient of x) and then adding the product to d, the equation reduces to the form as in R.39 and so, is to be solved by that rule, after that.

Ex.41. "Gaṇeśa (was worshipped) with 1 lotus flower. Śiva, Hari, Brahmā (and) the Sun (were worshipped) with 1/5, 1/4, 1/3 and 1/2 of what remained, successively. Kamalā was worshipped with the square-root of the lotus flowers (in the begnning) and the two feet of the teacher resembling lotus flowers, (were worshipped) with 1 lotus flower. Quickly say the number of lotus flowers in the collection."

EQUATION OF GUṆMAŪLA JĀTI

R.41(b)-42(a). "Multiply the multiplier of the root and the remainder by the multiplier. Solve (the equation so formed) by the method stated earlier. Divide the numbers so produced by the multiplier. (The quotient is the root)."[82]

The rule is meant for solving equation of the type,

$$x^2 \left(1-\frac{a}{f}\right) - \frac{b}{g} \cdot \left(\frac{c}{f}\, x\right) = d$$

Here, (c/f) is the multiplier of the root and d is the remainder. Both sides of the equation are multiplied by (c/f). Now, (cx/f) is taken as the quantity in place of x. This leads to another equation of the type stated in rule 39. Afters solving the equation by that rule, the division leads to the value of x.

Ex.42. "Seeing a swan with golden dazzling white wings in the garden, 7/8 of a group of ladies went hither and thither. $(3-\frac{1}{4})$ of the square-root of $\frac{1}{6}$ (of the group) attempted to catch the swan. Damayanti (the remaining lady) moving slowly with sweet melodious sound caught hold (of it). How many (of the ladies were there in the group?"

Ex.43. "3½ times ' the square-root of thrice (the number' of) black bees in a swarm went away from the swarm towards a part (of the lake) having lotus (flowers) and 3 went to *Kutaja* plants. 3 black bees) are still there. Tell quickly, how many black bees were there in the swarm."

EQUATIONS OF *HĪNAVARGA* AND *AṂŚAVARGA JĀTI*

R.42(b)-43. "The divider in one's own part is called *Rūpahara*. (This) multiplied by twice the subtractive is added to the square of *rūpahara*. (The result) happens to be the sum. The square of the subtractive is added to observed quantity (i.e., added to *dṛśya*). (The sum) multiplied by the square of rūpahara is the product of two numbers. (The numbers should be obtained) by the method stated earlier"[84]

The rule is meant for solving equations of the type

$$x - \left(\frac{n}{m} x - h\right)^2 = d.$$

Here, $\frac{1}{n/m}$ $(= \frac{m}{n})$ is *rūpahara*, h, the subtractive and d, the observed quantity. According to the rule, $2(m/n)h + (m/n)^2$ is the sum and $(m/n)^2\,(h^2+d)$ is the product (of the roots). From them, the roots are to be otained by rule 35 b The rule is obvious.

Ex.44. "On being challenged by wrestlers in the assembly of the king of

Matsya, Bhīma pressed the square of '1/4 of wrestlers less 6' under his feet, over powered 2 under his thigh and pressed the neck of 1. Tell how many of them were there.

In this example, Nārāyaṇa rejects one of the roots "due to irrelevancy."

Ex.45. "The square of 1/20 of a swarm of swans less 1 flew away in sky towards Mānasarovara and 95 (remained there). O friends, (tell) how many (of them were there)."

Ex.46.The square of 1/10 of a swarm of black bees are humming like young swans making sweet sound (while) eagerly looking for water (in their) abode and 24 of which went away to eat lotus having the smell of *Jambu* (tree). Tell), how many of the black bees were there in the swarm."

In examples 45 and 46, Nārāyaṇa considers both the roots as answers.

EQUATIONS OF BHĀGA SAṀGUṆYA JĀTI

R.44(a). "The observed quantity and unity, divided (severally) by (the product of) the fractions are the sum and the product respectively."[85]

The rule is meant for solving equations of the type, $x - (\frac{n}{m}x).(\frac{p}{q}x) = d$. According to the rule,

$\frac{1}{(\frac{n}{m}\frac{p}{q})}$ is equal to the sum and $\frac{d}{(\frac{n}{m}\frac{p}{q})}$, to the product of the rots.

Now, the roots of the equations can be obtained by a process similar to rule 35. The rule is obvious.

Ex.47. 1/6 of a group of monkeys multiplied by 1/12 of (them) are walking on the table-land (and)16, seeing the waterfall in the low-land, became eager for play. O friend, (tell) how many of them were there (in the group)."

EQUATIONS OF BHINNA-SAṀDṚŚYA JĀTI

R.44(b). "Unity less the observed fraction' divided by the product of fractions is the number."[86]

The rule is meant for solving equations of the type.

$$x - \frac{n}{m}x.\frac{p}{q}x = \frac{r}{s}x.$$

Ex.48. "1/6 of a group of large black bees multiplied by 1/9 of these, being afraid of the noise of a swarm of cuckoos on *Sāla, Tāla, Dala* and *Kadamba* trees, came to lotus (flowers) and 1/3 of the group came to *Pātalı* (tree). How many (of them were there in the group)?"

DOUBLE EQUATIONS OF THE SECOND DEGREE

R.45. "Any assumed number is the first number. Half the square of that is the

second (number). The sum and the difference of the squares of these (numbers), added to unity, became capable of yielding a square- root."[87]

According to the rule, $x = m^2/2$, $y = m$ is a solution of $x^2 + y^2 + 1 = u^2$ and $x^2 - y^2 + 1 = v^2$, where m is any assumed number. Rational of the solution has been given by Datta and Singh.[88]

Ex. 49. "The sum and the difference of squares (of two numbers), added to unity, are square. If you know (somethings), surprising on square, tell them quickly in many ways,"

R. 46. "The cube of any assumed number is the first numbers; 'a half of the square of its square' added to unity is the other (number). The sum and the difference of the squares of these two (numbers) less unity, happen to be squares."[89]

According to the rule, $x = (m^4/2)+1$, $y = m^3$ is a solution of $x^2 + y^2 - 1 = u^2$,

$x^2 - y^2 - 1 = v^2$, where m is any assumed number. Rational of the solution has been given by Datta and Singh.[90] The solution was found later on by E. Clere (1850).[91]

Ex.50. "The sum and the difference of squares (of two numbers) less 1, are squares. If you are curious about square, tell them quickly in many ways."

R.47. "Set down the square of the square of any assumed number at two places. Add the square (of the number at one place and) subtract (it at the other and) multiply (them) by 2. Their sum, (their) difference and 'their product added to unity', happen to be squares."

According to the rule, $x = 2(m^4 + m^2)$, $y = 2(m^4 - m^2)$ is a solution $x + y = u^2$, $x - y = v^2$, $xy + 1 = w^2$, where m is an assumed number. Rational of the solution has been given by Datta and Singh.[92] The rule was given earlier, as an example, by Bhāskara I.[93] Brahmagupta has given another solution for the problem.[94]

Ex.51. "The sum, the difference and 1 more than the product of two numbers are squares. If you have sufficient practice in 'curiosity about square', tell them."

R.48. "The sum of the squares of two assumed numbers happens to be the first (number). Twice their product (is) another number. Their sum and, difference taken separately happen to be squares."

According to the rule, $x = m^2 + n^2$, $y = 2mn$, is a solution of $x + y = u^2$, $x - y = v^2$ where m and n are any two assumed numbers.

Ex. 52. "The sum and the difference of two numbers are squares. O friend, if you have laboured in squares for this, tell what those two numbers are."

R. 49. "Divide the square of the cube of an assumed number by the square of the product of those two (numbers) stated earlier and multiply (the quotient) by those two numbers, separately. The sum and the difference (of the products) happen

to be squares and the product (of the products), a cube."

According to the rule, $x = (m^2 + n^2)p^6/[2mn(m^2+n^2)]^2$, $y = 2mnp^6/[2mn(m^2 + n^2)]^2$ is solution of $x + y = u^2$, $x - y = v^2$, $xy = w^3$.

Rational of the solution has been given by Datta and Singh.[95]

Ex.53. "The sum and the difference of two numbers happen to be squares and (their) product, a cube. O friend, if you know, tell them to me."

R.50. "The square of the cube of an assumed number divided by 25, is one number and twice of it is the other. The sum of their squares happens to be a cube and the sum of cubes, a square."

According to the rule, $x = m^6/25$, $y = 2m^6/25$ is a solution of $x^2 + y^2 = u^3$, $x^3 + y^3 = v^2$,

where m is any assumed number. Rational of the solution has been given by Datta and Singh.[96]

Ex.54. "O friend, if (you) are an expert in 'curiosity about square' tell quickly those two numbers, the sum of whose cubes happens to be a square and the sum of squares, a cube."

R.51. "A number, kept at two places, (when) multiplied by two numbers (separately and then) added to 1, happens to be squares. The sum (of the two numbers) multiplied by 8 (and then) divided by the square of their difference, is the number."

According to the rule, $x = 8(a + b) / (a - b)^2$ is a solution of $ax + 1 = u^2$, $bx + 1 = v^2$

Rational of the solution has been given by Datta and Singh.[97]

Ex.55. "O learned, tell me that number which (when) separately multiplied by 5 (and) 3 (and then) added to 1, becomes capable of yielding a square-root."

Ex.56. "O expert in 'curiosity about square', tell which number, separately multiplied by 4 (and) 7 (and then) diminished by 3, becomes capable of yielding a square-root."

R.52. "A number, increased by another or decreased by still other, becomes a square. The sum of those two (latter) number is divided by an assumed number and (then) diminished by it an (after that) halved. The square of the result, added to the subtrahend, is the number."

According to the rule,

$$x = \left[\frac{1}{2}\left(\frac{a+b}{m} - m\right)\right]^2 + b,$$

is a solution of $x + a = u^2$, $x - b = v^2$.

The rule was earlier given by Brahmagupta.[98] Rational of the solution has been given by Datta and Singh.[99]

Ex. 57 "Which number, kept at two places increased by 17 (at one place or) diminished by 4 (at the other) becomes a square. O eminent mathematician, if (you know 'square', tell that quickly."

R.53. "(A number), added to 2 optional numbers (separately), becomes a square (each time). Their (i.e.,2 optional numbers') difference is added to 1 (and then) halved. (The result) is multiplied by it self, (the product), diminished by the greater (optional) number, happens to be the number."

Let $a > b$. According to the rule,

$$x = (\frac{a-b+1}{2})^2 - a$$

is a solution of $x + a = u^2, x + b = v^2$.

Ex. 58. "A number, separately added to 3 and 10, becomes capable of yielding a square-root (in each case). O friend, if you are an expert in 'square', tell me that (number)."

R.54. "Two optional numbers (when) subtracted from (another number, separately) make it a square (each time). 1 is subtracted from the difference (of those 2 optional numbers and then the resulting difference) is halved. (The result) is multiplied by itself. (The product) increased by the greater (optional) number happens to be the (other) number."

Let $a > b$. According to the rule,

$$x = (\frac{a-b-1}{2})^2 - a$$

is a solution of $x - a = u^2, x - b = v^2$.

Ex. 59. "O friend, if you are an expert in the method of *pāṭī (gaṇita),* tell that number which, (when separately) diminished by 27 (or) 4, (becomes) capable of yielding a square - root."

R. 55. "An arbitrary number is the first number. The square of that less one-fourth (the square) is divided by an assumed number and then diminished by it and (also), by the first. A half of the remainder is the other (number). The sum of squares (of the two numbers) increased by their product is a square."

Let m and n be any two assumed numbers. According to the rule,

$$x = \frac{1}{2}\left(\frac{3m^2}{4n} - n - m\right), y = m,$$

is a solution of the equation,

$$x^2 + xy + y^2 = u^2.$$

Ex.60. "The sum of the squares of any two numbers increased by (their) product, become capable of yielding a square-root. Tell them, quickly, (if you) have interest in 'curiosity about square."

Nārāyaṇa states : "The rest (of the rules on the topic) is useful in (determinations pertaining to) area and so the same is being told thereat."

EQUALITY OF PRODUCT, SUM ETC.

R.56. "Keep the sum of two assumed numbers at two places. Those divided by the assumed numbers (separately, are) the numbers. Their sum is determined by their product."

Let m and n be any two assumed numbers.

According to the rule, $x = (m+n)/m$, $(m + n)/n$

is a solution of $x + y = xy$.

Ex.61. "If you have laboured hard in this work tell quickly those two (numbers) in many ways sum is equal to (whose) product."

R.57. "The sum of two assumed numbers multiplied (by them, separately) and (then each) divided by the sum of their squares are the two numbers, whose sum happens to be equal to the sum of whose squares."

Let m and n be any two assumed numbers.

According to the rule, $x = m\,(m+n) / (m^2+n^2)$, $y = n\,(m+n) / m^2+n^2)$

is a solution of $x + y = x^2 + y^2$.

Ex. 62. "The sum of squares of two (numbers) is equal to (their) sum. O friend, if you are an expert, tell them quickly in many ways."

R.58. "Divide the sum of squares, square of the sum and the product of any two assumed numbers by the sum of (their) cubes and the cube of (their) sum and (then) multiply by the two numbers. (The results will) be the two numbers, the sum of (whose) cubes and the cube of (whose) sum will be equal to the sum of their squares, the square of (their) sum (and their) product."

Let m and n be any two assumed numbers. According to the rule,

(1.1) $x = (m^2 + n^2)\, m/(m^3 + n^3)$, $y = (m^2 + n^2)\, n/(m^3 + n^3)$

(1.2) $x = (m + n)^2\, m/(m^3 + n^3)$, $y = (m + n)^2\, n/(m^3 + n^3)$

(1.3) $x = m^2\, n/(m^3 + n^3)$, $y = mn^2/ (m^3 + n^3)$

(1.4) $x = (m^2 + n^2)\, m/(m + n)^3$, $y = (m^2 + n^2)\, n/(m + n)^3$

(1.5) $x = (m + n)^2\, m/(m + n)^3$, $y = (m + n)^2 n/ (m + n)^3$

(1.6) $x = m^2\, n/(m + n)^3$, $y = m.n^2/(m + n)^3$

are the solutions of the equations

(2.1) $x^3 + y^3 = x^2 + y^2$

(2.2) $x^3 + y^3 = (x + y)^2$

(2.3) $x^3 + y^3 = xy$

(2.4) $(x + y)^3 = x^2 + y^2$

(2.5) $(x + y)^3 = (x^2 + y)^2$

(2.5) $(x + y)^3 = xy$, in order

Ex. 63. "Tell those two numbers, the sum of whose squares, the square of (whose) sum and (whose) product are (severally) equal to the sum of whose cubes (and), to the cube of whose sum".

Rule of Inversion

R.59. "(Beginning with) the visible number (working backwards), interchange the square (and) the square-root; the divisor (and) the multiplier (and) the positive (and) the negative. (In case) of one's own proportionate part is more or less, (the observed number) divided by 'unity increased or diminished (respectively) by the proportionate part' is the (desired) number."[100]

Ex.64. "O friend, if you are skilled and have considerable practice in mathematics, tell that number quickly, which (when) multiplied by 4, (the product) diminished by 3/4 (of it), (the difference) increased by 6, (the sum) multiplied by it self, (the product) divided by 60, (the quotient) increased by 2/3 (of it and then) the square-root (of the sum, when) diminished by 2, the remainder is unity."

Rule of Three

R.60. "*Pramāṇa* (i.e., the argument) and *icchā* (i.e, the requisition) are of the same denomination (and they should be set down) in the first and the last places. *Phala* (i.e., the result) is of a different denomination (and it should be set down) in the middle. That (placed in the middle) multiplied by the last and divided by the first happens to be icchāphala (i.e., the desired result)."[101]

Ex.65. "8 coconuts are available for 5 *paṇas*. O friend, tell me the price of 40 fruits."

Here, argument = 8 fruits, fruit (i.e., result) = 5 *paṇas* and requisition = 40 fruits, Writing these as directed in the rule,

8 5 40

Applying the rule, the desired result

$$\left(=\frac{40 \times 5}{8}\right) = 25 \textit{ paṇas}.$$

Ex.66. "8½ *palas* of *kumkuma* are obtained for 3¼ *panas*. Tell how much *kumkuma* will be obtained in $9\frac{5}{6}$ *panas*."

Ex.67. "$8\frac{1}{3}$ *palas* of *guggula* are obtained for 3½ *panas*. O friend, tell how much of that will be obtained in 100 palas and 1 guñjā."

Ex.68. "3 kuḍava les 3 pādikās (of a substance) are obtained for $3\frac{9}{10}$ drammas. Tell quickly, what is the price of 1 *kuḍava* more than 1 khārī of that (substance)."

Ex.69. "A female seprent moves 1 cubit and 3½ aṅgulas in ½ a day. In how much time will it move 5½ *yojanas*."

INVERSE RULE OF THREE

R.61. "In cases of living beings regulated by their age, gold compared by their weight and touch and also ricecorn etc. (measured by) *kuḍava* etc., when there is a change in the unit of measurement, the 'inverse rule of three' is to be used."[102]

Inverse rule of three is to be used where with increase of the requisition, the fruit decreases or vice-versa. In inverse rule of three, the middle quantity multiplied by the first quantity and divided by the last quantity gives the result.

Ex.70. This example is the latter half of example 64.

Ex.71. "The wages (for which) a wanton lady of 16 years of age who looks attractive due to gestures, jestfullness and amorous nature, is obtained in 10 niṣkas. O learned mathematician, tell quickly, how much fee a sensualist will have to pay for a lady of 20 years of age."

Ex.72. "An ox who can carry burden in a yoke of 2 is obtained in 8 *niṣkas*. O friend, tell in how many *niṣkas* that, who can carry burden in a yoke of 5 will be obtained."

Ex. 73. "A quantity of gold amounts to 300 *suvarṇas* (when) measured by a *māṣa* of 6 *guñjas*. O freinds, tell how much that happens to be (when) measured by a *māṣa* of 5 *guñjas*."

Ex.74. "The measure (if a substance when measured) by a Kuḍava of 16 pādikās is 600 khāris. What (will be the measure of the substance when measured) by a *kuḍava* of 18 pādikās."

RULE OF FIVE ETC.

R.62. "Having exchange the sides of the fruit and the divisor, the product of larger set of numbers divided by the product of smaller set happens to be the (desired) fruit in the Rule of Five etc.[103]

Ex. 75. "The interest of 100 for 1 month is 5. Tell quickly, what is the interest

of 65 for 1 year. O learned, if you know *pāṭī (gaṇita)*, tell the principal from the interest (and the time and) time from the principal (and) the interest."

Ex.76. "The interest of 99¾ for a fortnight is 2½. If you know *Pāṭī (gaṇita)*, tell the interest 66½ for 10½ months."

Ex.77. "A labourer gets 7 *paṇas* (by working) for 3 days. How much 6 person will get (by working) for 10 days."

Example on Rule of Seven

Ex.78. "A Sheet of cloth of length 9 cubits and breadth3 cubits is available in 10 puraṇas. How much, 3 sheets, (each) of length 12 cubits and breadth 5 cubits will cost?"

Example on Rule of Nine

Ex.79. "A piece of timber whose breadth, length and thickness are 2, 9 and 1 cubits (respectively) is obtained in 8 drammas. If you are of clear intellect and know *pāṭī (gaṇita)*, tell the price of 2 pieces of timber of breadth, length and thickness, 3, 12 and 2 cubits (respectively)."

Example on Rule of Eleven

Ex. 80. "6 paṇas are obtained by a cart in drawing 1 log of length, breadth and thickness, 15, 3 and ½ cubits (respectively) upto 5 *krośas*. O friend, if you know *pāṭī (gaṇita)*, tell quickly, what is the cartage of 10 poles (whose) dimensions are (each) 3 more (than those stated earlier) in bringing (them) from 8 krośas."

Barter of Commodities

R.63 (a). "In barter of commodities, having inter-changed the prices and one's own divisor, the method stated earlier (i.e., the rule of five etc.) is applied."[104]

Ex. 81. "300 pomegranates are obtained for 1 *dramma* and 95 mangoes for 1 paṇa. O friend, if you are proficient in barter, (tell) how many mangoes will be had for 10 pomegranates."

Sale of Living Beings

R.63. (b). "In the sale of living beings, having interchanged the ages in the prescribed (sides), apply the rule stated earlier."[105]

Ex.82. "2 ladies of 16 years of age, are obtained for 60 niṣkas. O learned mathematician, tell what is the price of 6 ladies (each) of 20 years of age, with beautiful eyes and slow movement due to heavy hips."

REFERENCES

1. Dvivedi, Padmakara (Ed.), *Gaṇita Kaumudī* by *Nārāyaṇa Paṇḍita*, Benares, Part I, 1935; part II, 1942 (=GK).

2. Colebrooke, H.T. (Trans.), *Līlāvatī (by Bhāskara II)*, Kitab Mahal, Allahabad, 1967 (=L), 1-145.

3. Abyyankar, S.K., *Bhāskarācārya's Bījagaṇita and Its English Translation*, Bhāskarācārya Partishthana, Poona, 1980, 17-26.

4. Misra, Babuaji (ed.), The *Siddhānta Śekhara of Śrīpati* (=SŚ) University of Calcutta, Calcutta, Part II, 1947, 134.

5. Jain, L.C. (Ed. & Hindi Trans.), *Mahavirācarya's Gaṇita Sāra Saṁgraha*, Jain Saṁskṛiti Saṁrakshaka Saṁgh, Sholapur, 1963 (=GSS), 49-67.

6. Cf. Shukla, K.S., *The Pāṭīgaṇita of Śrīdharācārya* Department of Mathematics and Astronomy, Lucknow University, 1959 (=PG), 2-3. Also Sinha, K.N., "Śrīpati's Gaṇita Tilaka: English Translation with Introduction", *Gaṇita Bhāratī* (=GB), 4(1982), (=GT), 114.

7-8. Gupta, R.C. 'Decimal Denominational Terms in Ancient and Medieval India', *GB*,. 5 (1983), 9.

9. *PG*, 3; *GT*, 114; *L*, 1 (foot note).

10. PG, 3; L, 2.

11-12. *GSS*,4; *PG*,4 GT, 115; L, 2.

13. L, 2.

14. PG, 4; L, 2-3.

15. PG, 4.

16. Kapadia, H.R. (Ed.), *Gaṇita Tilaka (with the commentary of Siṁha Tilaka Sūri;* Gaikbad sanskrit series, Baroda, 1935 Appendix II, 104-105.

17. *GT*. 115-116; L, 5.

18. Datta, B.B. and Singh, A.N., *History of Hindu Mathematics*, Lahore, Part I (1953), Part II (1938), (=HHM), 130-133.

19. PG, 5-7. Dvivedī, Sudhākara (Ed), Triśatikā by Śrīdharācārya Benares, 1899 (=Tri.), verse (=vs) 1-4.

20. HHM, I, 136-43.

21,22. HHM, I, 143.

23,25. HHM, I, 149.

24. HHM, I, 146-47.

26. Sharma, R.S., (Ed. with S. Dvivedi's Com., Vāsanā, and Hindi Trans.), *Brāhma-sphuṭa-siddhānta of Brahmagupt*, 4 vols., New Delhi, 1966 (=BSS) chap. xii, verses (=vss) 55-56.

27. Dvivedi, Sudhākara (ed. with his own com.),. *Mahāsiddhānta of Āryabhaṭa* II, Benares Sanskrit Series, Benares, 1910, chap. xiv (=MS), vs 3.

28. *GSS*, 9; *PG*, 7; *GT*, 116 and *SŚ*, 3; *L*, 6; *Tri*, vss. 5-7.

29. Cf. *GSS*, 12; *PG*, 8; *Tri.*, vs. 9; *MS*, vs. 4; *L*, 8; *GT*, 116; *SŚ*,6.

30,32. *HHM*, I, 153-54.

33. *GSS*, 13; *PG*, 9; *Tris.*, *vs*. 11 (a) (i); *MS*, vs. 6; *GT*, 117; *L*, 9.

34. Cf. *GSS*, 13-14; *PG*, 8; *Tri*; *vs*. 10; *GT*, *117*; *L*, 9.

35. Cf. GSS, 13; PG, 9; Tri., vs. 11b; GT, 117,; L, 9.

36. Cf. GSS, 13; PG, 9; Tris., vs., 11 (a) (ii).

37. HHM,I, 156-160.

38. Cf. GSS, 15;PG, 9; Tris. vss. 12-13; MS, vss. 6-7; GT, 117; SŚ, 9; L, 11.

39. PG, 10. See also, HMM, I, 170-74.

40,41. HHM, I, 174-75.

42. Shukla, K.S., and Sarma, K.V. (ed. and trans.), *Āryabhaṭīya of Āryabhaṭa*, Indian National Science Academy, New Delhi, 1976, (= Ā), 35; GSS, 16;PG, 11; Tri., vs. 15(a) (ii); GT, 118; SŚ, 7; L, 12.

43. Cf. *BSS*, chap. xii, vs 6; *GSS*, 17; *PG*, 11; *Tri.*, vss. 14-15 (a) (i); *GT*, 117; L, 12.

44. Cf. *PG*, 11; *Tri.*, vs. 15(a) (ii); GT, 118.

45. Cf. *GT*, 118; *L*, 12

46. Cf. *L*, 12-13.

47. *PG*, 11-12.

48. Cf. Ā, 37; *BSS* chap. xii, vs 7; *GSS*, 18-19; *PG*, 12-13; *Tris.*, vss. 16-18; *MS*, vss. 8—10 (a); GT, 118; *SŚ*, 10;L, 14.

49. *PG*, 13-14. See also, *HHM*, I, 176.

50,51. *HHM*, I, 184.

52. Shukla, K.S, "Hindu Mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭiya* (I)", Gaṇita, 22, No. I, 1971, 118.

53. Cf. *BSS*, Chap. xii, vs. 2 (a); *GSS*, 48;*PG*, 17; *GT*, 121; *L*, 16.

54. Cf. *PG*, 15; GT, 118; *SŚ*, 11;*L*, 19.

55. Cf. *BSS*, chap. xii, vs. 8; *GSS*, 59; *PG*, 18; *SŚ*, 14;*L*, 17.

56. Cf. *BSS*, chap. xii, vs. 9 (b); *GSS*, 61; *PG*, 19; Tri., vs. 24(a); GT, 121;*SŚ*, 14;L, 18.

57. Cf. *GT*, 121-22;L, 18.

58. *HHM*, 188-191.

59. *PG*, 17-18.

60. Cf *BSS*, chap. xii, vs 2 (b) ; PG, 15; *Tri.*, *19; MS*, 14; *GT*, 118-119; *SŚ*, 11;L, 19.

61. Cf. *BSS* chap. xii, vs 3 (b); GSS, 36; PG, 16; *Tri. 20 (a); MS*, 15(a); *GT*, 119; SŚ, 12;L,20.

62. Cf. *BSS*, chap.xii, 4; *GSS*, 37; PG, 16; *Tri.*, 20 (b) ; *MS*, 15(b);*GT*, 119;*SŚ*, 12;*L*,20.

63. Cf. *BSS*, chap.xii, vs.5; *GSS*, 38; *PG*, 16-17; *Tri.* 21-22; *MS*, 16-17: *GT*, 119-20;*SŚ*, 12;*L*, 21

64. Cf. GSS, 6-7;PG, 7-8; Tri, vs. 8;L, 21-22.

65. Shukla, K.S. (Ed.), *Nārāyaṇa Paṇḍita's Bījagaṇitāvtaṁsa*, Part-I, Lucknow, 1970, 5-7.

66. *BSS*, chap.xvii, vss. 33-35.

67. Abhyankar, S.K., ref. no. 3 above, 9-10.

68. Cf. *BSS*, chap. xviii, vs. 37 a(i); *GSS*, 91; *MS*, 21 (b); *SŚ*,103;*L*, 29-30.

69. Cf. *BSS*, chap. xviii, vss. 37a (ii)- 38; *GSS*, 91; *MS*, 22, *SŚ*, 103;*L*,30.

70. Cf. *BSS*, chap. xviii, vs 97.

71. See also, *Ā*, 67.

72. See also, *GSS*, 217.

73. Cf. *BSS*, xvii, vs. 98. See also, GSS, 216.

74. Cf. L, 25-26. See also, *GSS*, 69;PG, 59-60;MS, vss, 19-20; *GT*, 122-24.

75. *GSS*, 69-73;*PG*, 59-61.

76. *GT*, 123-24; *L*, 27-29.

77-80. *HHM*, 59-63.

81. Cf. *GSS*, 74; *GT*, 124.

82. Cf. *GSS*, 76; *PG*, 64-65.

83-84. Cf. *GSS*, 77-80; *GT*, 126-27.

85. Cf. *GSS*, 78.

86. Cf, *GSS*, 81;GT, 126.

87-89. See also, *L*, 31-33.

88. *HHM*, II, 272-74.

90-91. *HHM*, II, 271.

92,94,95. *HHM*, II, 284-85.

93. Shukla, K.S. (Ed. & Trans.), *Laghu Bhaskarīya*, Department of Mathematics and Astronomy, Locknow University, 1963, Trans. 99.

96. *HHM*, II, 278-79.

97. *HHM*, II 262-66.

98-99. *HHM*, 259-60.

100. Cf. *A*,71; *BSS*, chap.xvii, vs. 18;*GSS*, 163;*PG*, 65;*MS*, vs. 23;*GT*, 127-28; *SŚ* 19; *L*, 24.

101. Cf. *A*, 68; *BSS*, chap. xii, vs. 10;*GSS*, 83;*PG*, 22-23;*Tri.*, vs. 29;*MS*, vs. 24-25 (a); *GT*, 128; *SŚ*, 20;*L*, 38-39.

102. Cf. *BSS*, chap. xii, vs. 11 (a): *PG*, 25;Tri., vs. 30 MS, 25(b); *SŚ* 20; L, 38-39.

103. Cf. *BSS*, chap. xii, vs. 11 (b)-12 : *PG*, 25;*L*, 38-39; *GSS*, 88; *Tri.*, vs.31; *GT*, 129-30;*SŚ*,22

104. Cf. *BSS*, chap. xii, vs. 13(a) ; *PG*, 28,*Tri.*, vs. 32 (a), *MS*, vs. 28(a); *GT*, 131; *L*,47.

105. Cf. *PG*, 29; *Tri.*, vs. 32(b); *GT*, 131 *SŚ*, 23.

CHAPTER II

Second chapter begins with a rule on partnership [R1(a)]. Purchase and sale are the next topics of treatment [R.1(b)- R.3;R.34(b)- R. 35 and R. 36(b) - R. 38]. After that the author deals with interest (R. 4.-R.9) and payment in instalments (R.10-R.17). He then treats the topic of alligation (R. 18-R.26). Present of gems and wealth is the next topic of treatment [R.27- R. 34 (a)]. The chapter ends while dealing with certain miscellaneous topics such as the cistern problem [R.36 (a)], meeting of two travellers [R 39-R. 40 (a)], wages paid from the commodity [R. 40 (b) - R. 42 (a)] and true and false statements [R 42 (b) - R 43 (a)].

DETERMINATION PERTAINING TO MIXTURE OF THINGS

PARTNERSHIP

R.1 (a). "The contributions severally divided by their sum (and) multiplied by the mixed amount, happen to be (the respective) fruits."[1]

Ex.1. "Initial amount (of certain persons) are 27, 45, 63 and 81 (and the total) return is 384. O learned, tell quickly their separate profits."

Ex.2. "30, 60,90 and 120*palas* of honey, ghee, milk and curd (in order) were thoroughly mixed and having bathed Śiva with that, the pots were filled (with the mixture. Tell) separately, the measures of honey etc. (in different pots), in *palas*."

Ex.3. "Four traders obtain 1000*paṇas* by tilling a land for 6 etc. days with 5 etc. horses, in order, O friend, tell, what are (their shares), separately."

RULE FOR PURCHASE AND SALE

R.1(b)-3(a). "The rate of purchase is the mixed (amount) and the purchase-price is also the mixed (amount). The sale-price is the principal and the rate of sale is also the principal. The sale-price less the purchase-price is the profit. The (unknown) fruit should be obtained from the knowns by the process of proportion and from the mixed (amount) arising from *rūpa* by the method of partnership, stated earlier."

Ex. 4. "Śali rice was purchased at the rate of 8 *khārīs* (per coin and) sold at the rate of 5 khārīs (per coin). The profit is 50. Tell the principal."

In this case, according to the rule, rate of purchase = 8 = mixed amount

and rate of sale = 5 = principal.

So, profit = 3. But total profit = 60. Therefore, by the rule of three, the desired principal = 100 and the mixed amount 160.

If either the rate of purchase or the rate of sale is unknown, these are to be found out by the rule of proportion.

Ex. 5. "*Śāli* rice, wheat and *kulmāṣa* pulse were purchased (in the proportion of) 3, 5 and 7 (per coin) for an amount of 69 (and then) sold (in the proportion of 1, 2 and 3 (per coin, in order). O friend, if the profits be equal or the profit added to or subtracted from the principals be equal, tell me the principal, separately."

Ex.6. "(Certain commodities) were purchased in (the proportion of) 6,7 and 8 (per coin) for a mixed amount of 217 and (then) sold in (the proportion of) 5, 6 and 7 (per coin). Say, how the principal minus the profit will be the same (in each case)."

R. 3 (b). "Multiplying the rate-prices (of the commodities by the corresponding proportions (in which the commodities are purchased and then) dividing (the product) by the corresponding rate-quantities, apply the rule stated earlier."[2]

For illustration of the rule see the solution of Ex. 9, given below.

Ex. 7. "1¼ *palas* of dry ginger are obtained for 1 *dramma*, $1\frac{1}{3}$ *palas* of asafoetida, for 2 drammas and $1\frac{1}{5}$ *palas* of long pepper, for 3 *dramma*. Tell me, the equal quantities (of commodities obtained for) 10 drammas."

Ex. 8. "2 *palas* of sandal-wood are obtained for 1 *niṣka* and 5 palas of saffron for 2 niṣkas. Give me 2 parts of saffron with 1 part of sandal-wood, for 10 niṣkas, for the worship of god."

Ex. 9. "Purchasing chick-pea, wheat and sesamum at the rate of 3, 4 and 5 (parts of these) for 2, 3 and 4 *niṣkas* (in order), equal quantities of these were sold at the rate of 6, 7 and 8 (parts of these) for 5, 6 and 7 niṣkas (in order). The sum of the profits was 293. If you have pride of knowledge and experience in mathematics, tell the profits for chick-pea etc., separately (and) also the principals,separately."

Solution : From the question, purchase prices of unit parts of chick-pea, wheat and sesamum will be obtained as 2/3, 3/4 and 4/5 and sale prices, 5/6, 6/7 and 7/8 in order. So, profits in selling will be 1/6, 3/28 and 3/40 and total profit, 293/840. but total profit is 293. Therefore, by R.1 (a) individual profits will be 140, 90 and 63 respectively. Also, if the profit is 293/840, the purchase prices are 2/3, 3/4 and 4/5. But total profit is 293. So, principals are 560, 630 and 672, respectively, by rule of three. Similarly, the amounts with profits will be obtained as 700, 720, and 735 respectively, by rule of three.

Ex.10. 95 pomegranate fruits are obtained for 1 *dramma* and 5 mangoes for 1 paṇa. O learned, tell, in what manner 93 fruits can be obtained.

Ex.11. 9 young swans are obtained for 5 lazy middle aged swans. Three young swans are obtained for $\frac{3}{5}$ th part of the swans (in the flock). The sum of the remaining (swans) and the young ones becomes 296. O friend, tell quickly and separately, how many swans and the young ones are (in the flock).

INTEREST

Ex.12."A sum lent at the rate of 5 percent per month amount to 108 in one year. O friend, tell me the principal and the interest, separately."

Ex.13. "A sum obtained at the rate of 2 *niṣkas* percent per month, was sublent at the rate of 5 percent per month. O friend, the profit in 10 months is 45. Tell, what is the principal."

Ex.14. "An amount of 100 was lent in 3 parts at the rate of 2, 3 and 4 percent per month for 5,8 and 20 months (in order). O best among wisemen, if the interests of the 3 parts or the interests added to or subtracted from the respective parts be equal, tell quickly, in what manner (this was possible)."

Ex.15. A rich man gave to a Brāhmaṇa amount, equal to twice of that (given) in the previous year, in each subsequent year for 5 years. The amount became 186. What was the amount (given) in the beginning?

R.4-5 (a). "An optional amount is divided by 'the difference or the sum of the interest (on that amount) subtracted from or added to the optional amount'. (The quiotient) happens to be the capital. The difference of the optional amount and the capital determines the interest. (In such cases) the sum or the difference of the interest added to or subtracted from the other capital are (separately) equal."

Let the interest of two capitals p and $\acute{p}$ lent at the rates of interest r and r' percent per month for times t and t' (in months) be I and $\acute{I}$, respectively. Also, let the interests of an optional amount n at the same rates and for the same times be i and $\acute{i}$, in order. According to the rule, if
$p = n/(n \mp i)$ and $\acute{p} = n/(n \mp \acute{i})$ ...(1)

Then $p \pm \acute{I} = \acute{P} \pm I$ .... (2), and

$I = P - n$ or $n - p$, and $\acute{I} = \acute{p} - n$ or $n - \acute{p}$, in order, according as + or - sign is taken in (1).

Rational; We have, $I = prt/100$ and $\acute{I} = \acute{p}\acute{r}\acute{t}/100$ ...(3)

and so, (2) leads to $p \mp (prt/100) = p' \mp (\acute{p}\, r'\, \acute{t}/100) = n$ (say)... (4), where n is an optional number.

Now, (2) and (4) leads to $I = p - n$ or $n - p$ and $\acute{I} = \acute{p} - n$ or $n - \acute{p}$, in order, according as + or - sign is taken in (2) and hence the 2nd part of the rule.

Also, we have, $i = nrt/100$ and $\acute{i} = n\acute{r}\acute{t}/100$ .....(5), Multiplying both the sides of (2) by n^2, with the help of (5) we get, $n\,(np \mp ip) = n\,(n\acute{p} \mp \acute{i}\acute{p})$, or $np/(n \mp i) = n\acute{p}/(n \mp \acute{i})$

and so, if p and p' are given by $n/(n\mp i)$ and $n/(n\mp i)$, respectively, both sides become equal and hence the 1st part of the rule,

Ex.16. "Two amounts were lent at the rates of 5 and 4 percent per month for 7 and 10 months, respectively. The interest on one added to or subtracted from the other capital is (separately) equal to (the interest on) the other added to or subtracted from the former capital. (Tell them)."

R.5(b)-7(a). "In two prescribed different sides, if the sum of either the interests or the capitals or the times or the capital and the time (of different sides) or the capital and the interest or the time and the interest (of the same side) is observed as mixed, their shares in the sum can be determined, separately, by the method of partnership, considering the products of two (known ingredients) in the two sides as contributions."[3]

At the same rate of interest, let the interest of two capitals P and p in times T and t be R and r, in order and $R + r = a$. Suppose that a and other 4 quantities are known, separately. Then, according to the rule R and r can be obtained, separately, with the help of R.1(a) (Rule for Partnership) by considering PT and pt as separate contributions and a as the mixed amount.

In fact, $R = PTa/(PT + pt)$, and $r = pta/(PT + pt)$

The rule is meant for finding the ingredients, separately, when any one sum out of $R + r$, $T + t$, $P+p$, $P + t$, $p + T$, $R + T$, $r + t$, $R + P$ or $r + p$ is known and the rest 4 ingredients are known, separately.

Rational : Since rate of interest is the same for both the sides, $R/r = PT/pt$. Also, $R + r = a$. Removing R and r, one by one, we have, $R = a\,PT/(TP + pt)$, and $r = apt/(PT + pt)$ Similarly, the other cases can be dealt with and hence the rule.

Ex.17. "The sum of interest of 100 in a month added to that of 60 in a year (at the same rate) is 41. Tell me the interests, (separately). (Also), tell (the ingredients) separately, from the mixed (quantity) by interchanging them in between those two sides."

R.7(b)-8. "In two prescribed different sides, when the sum of the interest on (one side) together with either the capital or the time on the other side (or the sum of) the capital and the time (both on the same side) is observed as a mixed quantity, the product of quantities on the larger number side is divided by the product of (quantities on) the other side. (The quotient) gives the product. The quantities can be obtained (separately) by the method stated earlier."

Let R and r be the interest of two capitals P and p in times T and t at the same rate of interest, respectively. Suppose that any one out of $P + r$, $p + R$, $T + r$, $R + t$, $P + T$ and $p + t$ is known and the rest 4 quantities are known, separately. Then the product of 2 quantities can be obtained from the given 4 quantities and then the quantities themselves can be obtained, separately, be rules 35 (b) and 31 of chap.I.

Rational : Since the rate of interest is the same, we have $R/r = PT/pt$ and so $Pr = ptR/T$ and similarly for the other cases and hence the rule.

Ex. 18. "The sum of the interest of 100 for a month and the capital whose interest in a year is 36, is 65. O friend, tell (them) separately. (Knowing) the sum of the capital and its time or that of the interest of 100 and the time (tell them, separately)."

R.9. "The capital multiplied by its own time and (then) divided by the other's time happens tobe the difference of two quantities. The same at the other place multiplied by the mixed quantity is the product. From them (the two quantities can be obtained separately). The positive (root) is the capital."

Let R be the interest of a capital P in time T and r, that of p in time t and also let $r + R = a$ Suppose that P, T, t and a are given. According to the rule, PT/t is 'the difference' and aPT/t is 'the product' of two quantities. The quantities themselves can be obtained separately by R.35(b) and R.31, chapter I. Positive root gives the value of R.

Rational : We have $R/r = PT/Rt$, and $R + r = a$. From these we obtain, $R^2t + RPT - aPT = O$. Now, this quadratic equation will have two roots, one negative and the other positive, the latter being smaller in magnitude and hence the rule.

Ex.19. "A rich man lent for 3 months, an amount equal to the interest of 150 for 4 months. The sum of the amount lent and the interest became 22. O friend, tell them, separately."

Payment in Instalments

R.10. "The capital is multiplied by the fixed period of instalment. (The product is) divided by 'the amount of instalment less the interest (on the total amount) for the period of instalment'. (The quotient). determines the time of being free from debt."

The author elaborates, "The fixed amount which the debtor gives to the creditor at the fixed interval is the amount of instalment. The fixed interval is the amount of instalment. The fixed interval is the period of instalment. Here always, the creditor first of all accepts interest on the principal."

Let the interest of a capital p in time t be i and let an amount A be lent to a debtor at this rate of interest. Also, let $\dot{p}$ be the agreed amount of instalment and $\acute{t}$, the period of instalment. Suppose that $\acute{i}$ is the interest on the total amount A for the period of instalment $\acute{t}$. If T be the time of being free from debt, according to the rule, $T = A\acute{t}/(\dot{p}-\acute{i})$.

Rational : Let I be the interest on amount A for time T. Since, the rate of interest is the same, we have $\acute{i}/i = A\acute{t}/pt$, and $I/i = AT/pt$ ...(1).

Also, total amount paid by the debtor $= \dot{p}T/\acute{t} = A + I$ ...(2).

These 3 relations leads to, $T = A\acute{t}/(\dot{p}-\acute{i})$ and hence the rule.

Ex.20. "In his bad days a debtor was given an amount of 90 at the rate of 5 percent per month to be paid back in biomonthly instalments of 15 (each). O friend, tell me quickly the time of being free from the debt."

R.11. "The time of being free from debt is multiplied by the amount of instalment. (The product is) divided by (the product of the sum of) the interest on

unity for the time of being free from debt added to I' multiplied by the period of instalment. (The quotient) is the capital."

Along with the notations of R.10, let $\acute{I}$ be the interest on unity for time T. According to the rule,

$$A = \dot{p}T/(1 + \dot{I})\acute{i}$$

Also from (1) and 2) under R.10, we get $A = \dot{p}T/t(1+i\,T/pt)$

But, $\acute{I}/i = 1.\dot{T}/pt$ and this leads to the desired result.

Ex.21. "An amount is lent at the rate of 5 percent per month. The period of instalment is 2½ months; the amount of instalment is 15 and the time of being free from debt is 30 months. If you know, tell the capital and the interest."

R.12-14(a) "The amount (lent) is divided by the amount of instalment. (The integral part of) the quotient is called *Pada* and the remaining part, *Agra*. *Pada* is added to its square. (The sum) is multiplied by half the amount of instalment. (The product) is added to (the product of) *agra* (i.e., the remaining part) multiplied by the amount (lent. The sum) is called *Mūla Piṇḍa*. The interest should be obtained from that (i.e., from the *mūlapiṇḍa*) by taking the time equal to the period of instalment. After that the period of instalment multiplied by the amount (lent, the product) divided by the amount of instalment happens to be the time of being free from debt."

In this case, at the agreed time of payment of instalment, the agreed amount of instalment towards the payment of the capital and total interest on the unpaid amount for the period of instalment are paid. With the notations of R.10, let, $A/a = n + f/a$ where a is the fixed amount of instalment to be adjusted towards the payment of the capital and n, an integer. The rule names n as *Pada*, f/a as *Agra* and $[(n^2 + n)a/2 + Af/a]$ as *Mūla Piṇḍa*. The rule states that the interest should be obtained by considering *mūla piṇḍa* as the amount lent and the period of instalment ($= t$) as the time. As in R.10, $\acute{i}/i = At/pt$ and so the interest $= [(n^2 + n)a/2 + Af/a]\ (it/pt)$.

Also according to the rule $T = (A/a)t$.

Rational : For the interest of intergral part of the capital, the debtor will have to pay the interest of n instalments (i.e., of amount na) for one period of instalment, of (n-1) instalment for one period and so on till the amount exhausts. Also he will have to pay interest due to fractional part for the total period. So, total interest $= [n+(n-1) + \ldots + 2 + 1]\ \frac{a\acute{i}t}{pt} + \frac{f\acute{i}T}{pt} = \left[\frac{n(n+1)a}{2} + \frac{Af}{a}\right]\frac{\acute{i}t}{pt}$, as $T = (A/a)t$ as from 2nd part of the rule. 2nd part of the rule is obvious.

Ex.22. "Somebody lent 75 at the rate of 5 percent per month. The debtor, for fear of (high) interest, returns (the amount) in instalments of 9 (along with the interest due), bimonthly. O friend, if you know, tell the interest and the time of being free from debt."

R.14 (b)-16(a). "Subtract the principal part of the instalment for (the respective) period of instalment from the capital, successively. (This gives the number of complete months and a residue of the capital, if any). Subtract the

interest of the residue for a month from the monthly amount of instalment. Multiply the interest of the residue for a month by the number of complete months. Add the residue of the capital to (the product). Divide (the sum) by the (above) remainder. The quotient added to the number of complete months happens to be the time of being free from debt."[4]

In this case the amount of instalment consists of the principal part (i.e., the present worth of the instalment) together with its interest for the period after which the payment is made. As in the rule, let m be the number of complete months clapsed and R, residue of the capital. also, let a be the amount of instalment and $\acute{I}$ be the interest of R for a month at the rate of interest at which the sum is lent. According to the rule, the time of being free from the debt (= T, say) = $m + (m\acute{I}+R)/(a-\acute{I})$

Rational : Let $T = m + k$, in months where $k < 1$. Since the amount of monthly instalment a consists of the principal part and its interest for the period after which the payment is made, we observe that, the ratio of $R + (m + k)\acute{I}$ and a will be the same as that of k and 1. So,

$$\frac{R+(m+k)\acute{I}}{a} = \frac{k}{1}.$$

This leads to $k = (m\acute{I} + R)/(a - \acute{I})$ and hence the rule.

Ex.23. A rich man gave (someone) 150 at the rate of 10 percent per month and the successful debtor gives a monthly instalment of 50. O friend, tell quickly by what time, the debtor will be free from debt.

R.16 (b)-17. "Substract the principal part of each monthly instalment free from interest from the capital, successively. (This gives the number of complete months and a residue of the capital, if any). Subtract the interest of residue for a month from the amount of monthly instalment. Divide the residue of the capital by the remainder (of the monthly instalment, the quotient) added to the number of (complete) months is the time (of being free from debt)."

In this case, after paying interest of the capital for the period of instalment, the rest amount of the instalment is adjusted towards the payment of the capital. Also, the interest is charged on the remaining capital. The process is repeated with the remaining capital. This gives the number of complete months (= m) and a residue of the capital (= $\acute{R}$). Let a be the amount of monthly instalment and $\acute{I}$ be the interest of $\acute{R}$ for a month at the rate at which the amount is lent. According to the rule, the time of being free from debt (=T, *say*) = $m + \acute{R} / (a - \acute{I})$.

Let $T = m + k$ in months where $k < 1$. Since the amount of instalment consists of the principal part together with the interest of the capital remaining after adjustment for the period of instalment, we observe that the ratio of $\acute{R} + k\acute{I}$ and a will be the same as that of k and 1. So, $(\acute{R} + k\acute{I}) / a = k/1$.

This leads to $k = \acute{R} / (a - \acute{I})$ and hence the rule.

Ex.24. "A rich man lent somebody 100 at the rate of 10 percent per month. (The debtor) gives a monthly instalment of 50 including the interest. O friend, tell the debtor's time of being free from debt."

NOW ALLIGATION

Varṇa is the colour of gold on the touchstone, i.e., fineness of gold determined by that touch. It is analogous (though not equivalent) to the modern term 'carat', gold of 1 *varṇa* contains 1 part of pure gold and 15 parts of impurities in the form of baser metals, gold of 2 *varṇas* contains 2 parts of pure gold and 14 parts of impurities and so on. Pure gold called *Kalyana-suvarṇa*, is of 16 *varṇas* (24 carates). The amount of pure gold in a piece of gold of weight w and *varṇa* v is equal to $wv/16$.

R.18. "Divide 'the sum of the product of weights of the pieces of gold multiplied by one's own touch' by the sum of the weights of pieces of gold. (The quotient) happens to be the touch (of the alloy). (In case of refining) considering the weight of the refined gold as (the sum of the pieces of) gold, (obtain) the weight of (the refined) gold from (its) touch (and vice-versa)."[5]

Let n pieces of gold of weight $w_1, w_2 .... w_n$ and touches $v_1, v_2,...., v_n$, respectively be melted together. Let v be the touch of the alloy. According to the rule,

$$v = \frac{w_1v_1 + w_2v_2 + ... + w_nv_n}{w_1 + w_2 +, ... + w_n}$$

In case of refining, if W be the weight and V, the touch of the refined gold,

$$V = \frac{w_1v_1 + w_2v_2 + ... + w_nv_n}{W} \quad \text{and}$$

$$W = \frac{w_1v_1 + w_2v_2 + ... + w_nv_n}{V}$$

Rational : Amount of pure gold in the above described n pieces of gold

$$= \frac{w_1v_1}{16} + \frac{w_2v_2}{16} + ... + \frac{w_nv_n}{16}$$

and the amount of pure gold in the alloy of these pieces $= \frac{(w_1 + w_2 + ... + w_n)}{16}$

and since the amount of gold in both the cases is the same, hence the rule. Case of refining can be similarly dealt with.

Ex.25. "Pieces of gold of weight 9, 11, 13 and 15 *māṣas* and of touches 2, 9, 6 and 3 (respectively) are melted together. O learned, if you have laboured in alligation, tell here quickly, the touch of the gold. If after refining that, the touch be 15, tell me the weight (of the gold and if) the weight of gold be 16 (tell the touch)."

R.19. "Subtract the sum of he product of weights of pieces of gold (of known touches) multiplied by their respective touches from the product of the sum of the weights of (all)the pieces of gold multiplied by the touch of their alloy. Divide (the difference) by the weight of the pieces of gold (of unknown touch). (The quotient) is the measure of the unknown touch of that (piece of gold)."[6]

Let n pieces of gold of weights w_1, w_2----,w_n and touches $v_1, v_2, ... ,v_n$, respectively, be melted

together along with another pieces of gold of weight w and of unknown touch V and let the touch of the alloy be $\acute{V}$. According to the rule,

$$V = \frac{(w_1 + w_2 + \dots + w_n + w)V' - (w_1 v_1 + w_2 v_2 + \dots (+ w_n v_n)}{W}$$

Ex.26. Pieces of gold of weights 3,2 and 1 *māṣas* and of touches 10, 12 and 14 (respectively) are melted with a piece of gold of 5 *māṣas* of weight and of unknown touch and the touch of the alloy becomes 13. O learned, tell quickly the touch of that (piece of gold).

R.20. "The sum of weights of pieces of gold is multiplied by the touch of the refined alloy. (Also) the products of the weights of pieces of gold (multiplied by their respective) touches are added. The difference of the (above) product and the sum is divided by the difference of the known touch of a piece of gold and the touch of the alloy. (The quotient) is the weight of the piece of gold."[7]

With the notations of R.19, the rule states that,

$$W = \frac{(w_1 v_1 + w_2 v_2 + \dots + w_n v_n) - (w_1 + w_2 + \dots + w_n) V'}{\acute{V} - V}$$

Ex.27. "(Pieces of) gold of touches 16,15 and 12 and weights, 2,3 and 4 (respectively, are melted together) with a piece of gold of touche 9. O friend, if you know *suvraṇagaṇita*, tell the weight (of the piece of gold), the touch of the alloy being 11."

R.21. "Leaving one and supposing the weights of other pieces of gold, the touch of the unknown piece of gold should be obtained by the method stated earlier."

Let the touches of n pieces of gold of unknown weights be v_1, $v_2, \dots, v_n$, then supposing the weights of the pieces as w_2, $w_3, \dots, w_n$ whose touches are v_2, v_3, ... , v_n, respectively, the unknown weight w_1 of the piece whose touch is v_1 can be obtained by R.20 and will be given by

$$w_1 = \frac{w_2 v_2 + w_3 v_3 + \dots + w_n v_n - (w_2 + w_3 + \dots + w_n) v}{v - v_1},$$

v being the touch of the alloy.

Ex.28. "Pieces of gold of touches 8, 10, 15 and 16 are melted together and the touch of the alloy is 14. O mathematician, if you have proficiency in *suvarṇagaṇita,* tell the measures of weights of pieces of gold in several ways."

Formation of Test sticks

R.22-23(a). "Dividing the difference of the touches of the superior and the inferior pieces of gold by the assumed fraction, (the quotient) is repeatedly and separately lessened by 1 until unity is obtained in the end. Remainders (and the same) reversd, are the weight of the inferior and the superior pieces of gold (respectively)."

According to Nārāyaṇa, "Here, the sum of weights of superior and inferior pieces of gold remains constant (and) the touch of the alloy remains in excess by an integral multiple of the assumed fraction from the touch of inferior pieces of gold."

Let t and t' be the touches of inferior and superior pieces of gold, respectively. Also, let W be their combined weight and w, the weight of inferior pieces of gold. Suppose that test sticks are formed such that their touches increase by a fraction n/f over the touch of inferior piece, n being an integer. If,

$$wt + (W - w)\, t' = Wt + Wn,$$

then according to the rule, $w = \frac{n(t'-t)}{f} - 1, \frac{n(t'-t)}{f} - 2$ etc; the weights of the superior pieces being in the reverse order.

Rational : From R.18, we have, $wt + (W - w)\, t' = W(t + n/f)$, as according to supposition, touch of the alloy is $t + n/f$. This leads to $w\,(t' - t) = \frac{Wn}{f}\,(\frac{t'-t}{n/f} - 1)$. Now, if $Wn/f = t' - t$, we get, $w = (\frac{t'-t}{n/f} - n)$. Putting $n = 1, 2, 3, \ldots$ we get the result.

Ex.29. "Two pieces of gold are of touches 12 and 16. From these test sticks (of weights 1 *māṣa* are formed whose) touches increase by 1/4 over that of the inferior piece. Tell (their weights and respective touches)."

R.23(b)-24 (a). "Multiply the touch of a piece of gold by the sum of their weights. (Also) multiply the touch of the refined gold by its weight. Divide the difference (of the above products) by the difference of touches (of the two pieces). (The quotient) happens to be the weight of the (other) piece (of gold and) vice-versa."

Let two pieces of gold of known touches t_1 and t_1 and unknown weights w_1 and w_2 respectively, such that $w_1 + w_2 = \acute{w}$, be melted together. Also, let the touch and weight of the refined together. Also, let the touche and weight of the refined gold be T and W, respectively. According to the rule,

$$w_1 = (WT - \acute{w}t_2) / (t_1 - t_2) \text{ and } w_2 = (\acute{w}t_1 - WT) / (t_1 - t_2).$$

Rational : From R.18, we have, $w_1t_1 + w_2t_2 = WT$. Adding w_1t_2 to both the sides and simplifying we get the first result. Similarly, the other result can be derived.

E.x30. "Two pieces of inferior gold of touches 10 and 14 and of total weight 55, are melted (together). The touch and the weight of the refined piece are 13 and 50. (respectively). Tell me, (their weights) separately."

R.24(b)-26. "The numerator is called *Khaṇḍa* and the denominator is identical with the weight of the piece of gold. Multiply one's own weight by (its) touch. (Multiply the product by) the sum of one's own *khaṇḍa* and the other's weight. (Again) multiply others's touch by one's own *khaṇḍa*. (Multiply the product by) the sum of other's *khaṇḍa* and one's own weight. Subtract the (latter) product (from the former). Divide the remainder by the excess of the product) of weights over (that of) *khaṇḍas*. Know (the quotient) as the measure of the other's touch. (Obtain) the touch of the second (piece) in the same manner."

Let a_1/w_1 of a piece of gold of weight w_1 and unknown touch T_1 when mixed with another piece of gold of weight w_2 and unknown touch T_2 produce a piece of gold of touch t_1. Also, let a_2/w_2 of the piece of gold of weight w_2 and unknown touch T_2 when mixed with the piece of gold of weight w_1 and unknown touch T_1 produce a piece of gold of touch t_2.

According to the rule,

$$T_1 = \frac{w_2t_2(a_2 + w_1) - a_2t_1\ (a_1 + w_2)}{w_2w_2 - a_1a_2}$$

and

$$T_2 = \frac{w_1t_1(a_1 + w_2) - a_1t_2\ (a_2 + w_1)}{w_1w_2 - a_1a_2}$$

Rational : By R.18, we have, $\frac{a_1}{w_1} \cdot W_1T_1 + w_2T_2 = (\frac{a_1}{w_1} \cdot w_1 + w_2)t_1$

Similarly , $\frac{a_2}{w_2} \cdot W_2T_2 + w_1T_1 = (\frac{a_2}{w_2} \cdot w_2 + w_1)t_2$

Multiplying both the sides of the former by a_2 and that of the latter by w_2 and subtracting the former produce from the latter product we get the desired value of T_1. Similarly the value of T_2 can be obtained and hence the rule.

Ex.31. "Out of 2 pieces of gold of unknown touches and weighrs (when) 2/3 of the first (pieces) is melted with total of the other (a piece of gold) of touch 12 is obtained (and when) 4/5 of the other mixed with total of the first the touch of (the piece of gold so produced) becomes 11. Tell, the touches and the weights of the two (pieces, separately)."

MISCELLANEOUS TOPICS

Present of Gems and Wealth

R.27. (a). "Multiply the number of person by the number of gifts. Subtract (the product) from the number of (any kind of) gems. Any optional number divided by (the difference) is the price (of a gem of that kind)."[8]

Let each among n persons possess several gems of a particular kind. Also, let every one of them make a gift of g gems to each of the other and thus become equally rich. If m be the number of gems possessed by a person, according to the rule, the price of a gem of that kind = $N/(m-ng)$, N being an optional number.

Rational : Since, m-ng is the same fore each person = N, say, therefore, the rule follows from it.

Ex.32. (Four persons possessed) sapphires, *gomedas vaidūryas* and diamonds numbering 5,6 7 and 8, in order. Everyone of them gave one of their gems to each other. (thus) the price (of gems in their possession) became equal. O friend, tell quickly the price of gems.

R.27(b). "An optional number divided by 'the multiplier less one' is the separate amount of wealth."

Let a number of persons possess wealth whose values are x, y, z.... and let $ax + y + z + ...$

$x + by + z + ... = x + y + cz + ...$ ($=k$, say). According to the rule, wealth of a person $= n/(m-1)$, where n is an optional number and m is the multiplier corresponding to that person, i.e., $m = a, b, c,..$ etc.

Rational : Since total wealth of persons are fixed, $x + y + z + ... = k - n$, say, n being an optional number. Subtraction of the former relation from this relation leads to the desired relation.

Ex.33. "Four persons. went to sea. One of them asked others to tell the price of the central gem of a necklace, their wealth (separately) and the sum of their wealth, if 3 times his wealth together with the sum of other's wealth (and also) 4,5 and 6 times the wealth of others (separately) together with the sum of the rest were equal to the price of the gem."

R.28. "The sum of the depleted amounts divided by 'the number of persons less one' is the total amount. The stated amounts are subtracted from that, severally. (The results) are the differents amount."[9]

Let n persons possess wealth whose values are $x, y, z, ...$ and let their combined wealth excluding that of that of one person in turn be $a, b, c,...$ etc. so that, $y + z + ... = a, x + z + ... = b, x + y + ... = c$ and so on ....(1). Let $a + b + c + ... = S$ .... (2).

According to the rule, $x + y + z + ... = S/(n - 1)$ and $x = \frac{S}{(n-1)} - a$; $y = \frac{S}{(n-1)} - b$, etc.

Rational : Adding respective sides in (1), we get,

$$(n - 1)x + (n - 1)y + (n - 1)z + ... = a + b + c ...$$

This leads to $x + y + z + ... = S/(n - 1)$, due to (2) which gives the first part of the rule. Second part of the rule immediately follows from this relation as $y + z + ... = a, x + z + ... = b$, etc.

Ex.34. "Horses of 4 persons were combined in a group. On being asked to tell their numbers, quickly and separately for toll, they told (that) the sum of their horses excluding their own as 15 for the first and 2 more (each time than that of the previous one) for others. O friend, (tell), how many horses they had."

R.29. "(One's) multiplier is added to 1. An optional number divided by the sum is the desired amount of wealth of the person (corresponding to the multiplier). Therefore, the optional number less their sum is, certainly, the amount of wealth.

Let an amount of wealth be w and let the wealth of m persons be $x, y, z,....$ Suppose that the amount of wealth added to the wealth of a person equals some multiple of the rest, the multipliers being $p, q, r,..,$ for $x, y, z, ..,$ in order. Also, as in R.25, let $x + y + z + ... = S$. Again, suppose that $w + S = n$, where n is an optional number. According to the rule, $n/(p+1) = n/(p+1) = a$, $n/(q+1) = b$, $n/(r+1) = c$ etc. and $w = n - S$.

Rational : From supposition, we have, $x + w = p(y + z + ...)$ Adding $y + z + ...$ to both the sides and then substituting S and a in place of its equivalent expression, we have, $w + S = (p + 1)a$. This gives $n/(p + 1) = a$. Similarly, other results can be obtained and hence the first part of the rule. Second part of the rule is obvious.

Ex.35. "Four person got a treasure. One of them told that if the treasure was

combined with his wealth (the sum) would be 4 times (the sum of) others. And, in this way, others told (that the sum would be) 5, 6 and 7 times (the sum of the rest). Tell me their wealth, separately."

CASE OF DISSIMILAR PURSE

R.30-32. "Add ones own fraction to the product of other's fractions with on's own multiplier (severally). Divide the sums by the corresponding (other's) multiplier as increased by 1. Add (the quotients). Multiply one's own fraction by 'the number of persons less 2'. Subtract the product from the *sum*. Divide the difference by one's multiplier as increased by 1. (The quotient) happens to be one's own wealth. (By doing so) again and again (the other's) wealth (can be obtained). Subtract ones own wealth from the product of the sum of other's wealth one's own multiplier. (The difference) divided by one's own fraction happens to be the amount of wealth. Alternatively, reduce the proper fractions to their common denominator. Remove the denominators. Those (wealth can be obtained) with such numerators (in place of fractions) by the method stated earlier."10

Let w be an amount of wealth. Let n persons possess wealth x, y, z, etc, their sum being s, i.e., $x + y + z + \ldots\ldots = s$.....(1). Suppose that when w is multiplied by different fractions, f, g, h etc. and then added to x, y, z, etc., in order , the sum becomes a multiple of the rest i.e., $wf + x = p\,(y + z + \ldots.)$; $wg + y = q\,(x + z + \ldots)$; $wh + z = r\,(x + y + \ldots.)$ and so on ....(2)

The rule states that,

$$x = \frac{1}{(p+1)}\left[(\frac{pg+f}{q+1}) + (\frac{ph+f}{r+1}) + \ldots - f(n-2)\right] \ldots..(3)$$ and similar expression for $y, z, \ldots$ etc.

and $$w = \frac{p(y+z+\ldots)-x}{f} = \frac{q(x+z+\ldots)-y}{g} = \qquad \ldots.(4)$$

Alternatively, reducing the fractions to a common denominator and then removing the denominators. one may proceed with the numerators in place of fraction.

Rational : Let $y + z + \ldots.. = a$, $x + z + \ldots. = b.$, $x + y + \ldots. = c$, etc...(5). Adding $y + z + \ldots$ to both the sides of the first expression in (2), due to relations (1) and (5), we get, $(s + wf) / (p + 1) = a$, similarly $(s + wg)/ (q + 1) = b$, etc.

Adding these, we have

$$S(\frac{1}{p+1} + \frac{1}{q+1} + \ldots) + w(\frac{f}{p+1} + \frac{g}{q+1} + \ldots) = a + b + = (n - 1)s, \text{ due to (5) and (1)}$$

or $$S = \left[\frac{w(\frac{f}{p+1} + \frac{g}{q+1} + \ldots)}{D}\right], \text{ where } D = (n - 1) - \frac{1}{p+1} - \frac{1}{q+1} - \ldots\ldots$$

or $$s - (y + z + \ldots) = \frac{w(\frac{f}{p+1} + \frac{g}{q+1} + \ldots)}{D} - \left(\frac{x + wf}{p}\right), \text{ from (2),}$$

or $$x = \frac{w\left(\frac{f}{p+1} + \frac{g}{q+1} + \ldots..\right)}{D} - \frac{x}{p} - \frac{wf}{p}$$

On solving for x and then putting $w = D$ we get the value of x as stated in the rule. Similarly for $y, z, \ldots$ and hence the first part of the rule.

It is clear that if $w/D \neq 1$, we must multiply the expression for $x, y, z, \ldots$ by w/D in order to get the correct values for $x, y, z, \ldots$ etc.

Second part of the rule is easily derivable from (2). Third part of the rule is obvious.

Ex.36. "5th, 6th, 7th and 4th parts of the obtained wealth added separately to the wealth of each of 4 persons become 4,3,2 and 5 times the sum of wealth of the rest (in order). (tell them, separately)."

R.33-34(a). "Multiply the sum of the amounts received (by a person) by '(the respective) multiplier as increased by 1' (Then proceed to obtain wealth of persons) as in the method for dissimilar purse. One's own multiplier is divided by 'the multiplier as increased by 1'.

The quotients are added. The sum is diminished by 1. (The wealth of persons as obtained above) is divided, separately, by the difference. (The quotients happen to be the real) wealth of those persons, one after another."[11]

Let $W, d, n, p, q, r, \ldots, x, y, z, \ldots$ and $f, g, h, \ldots$ *have* same meanings as attached to these in R 30-32. Also, let $f(y), g(z), h(x), \ldots$ denote the amounts of wealth received by persons whose wealth are $x, y, z, \ldots$ and given by persons whose wealth are $y, z, x, \ldots$, in order. According to the rule, first the (so called) values of $x, y, z, \ldots$ should be obtained by. R.30-32, supposing $(p+1)\,[f(y) + f(z) + \ldots] = f$; $(q+1)\,[g(x) + g(z) + \ldots] = g$ and so on and $D = \frac{p}{p+1} + \frac{q}{q+1} + \ldots - 1$. The real values of $x, y, z \ldots$ are then given by $x/D, y/D, z/D, \ldots$ in order.

Rational : Suppose that the persons possessing wealth $y, z, \ldots$ give $f(y), f(z), \ldots$ wealth to the person possessing wealth x, so that his wealth becomes p times those of the rest. Thus, we have, $x + f(y) + f(z) + \ldots = p\,[(y + z + \ldots) - f(y) - f(z) \ldots]$ Adding $y + z + \ldots$ to both the sides, since $s = x + y + z + \ldots$) we get after simplication, $\frac{s}{p+1} + [f(y) + f(z) + \ldots] = y + z + \ldots$ Now, if we suppose $(p+1)\,[f(y) + f(z) + \ldots] = f$, as proposed in the rule, this leads to $s + f = (p+1)(y + z + \ldots)$

or $x + f = p\,(y + z + \ldots)$ as $s - (y + z + \ldots) = x$. Now, if we compare this equation with equation (22) of R.30-32, after supposing $w = 1$, in that rule, we find that both the equations become identical and therefore making use of relation (3) of that rule, we have,

$$x = \frac{1}{(p+1)\,D}\left[\left(\frac{pg+f}{q+1}\right) + \left(\frac{ph+f}{r+1}\right) + \ldots - f(n-2)\right]$$

where, $$D = n - 1 - \frac{1}{p+1}\;\frac{1}{q+1} - \frac{1}{r+1} \ldots$$

$$= \frac{p}{p+1} + \frac{q}{q+1} + \frac{r}{r+1} + \ldots - 1$$

and similar expressions for $y, z, \ldots$ etc. and hence the rule.

Ex.37. "First out of 3 traders speaks with vigour that if both give him 6 and 8 his (wealth) will be twice that of their's (remaining wealth). Another (tells) that (if others) give him 8 and 7 (his wealth) will be thrice that of their's and the other

(asks to give) 7 and 9 and (his wealth) will be 5 times their wealth. If you have pride in mathematics, tell their wealth."

PURCHASE AND SALE (CONTAINED)

R.34.(b)-35. "Multiply the number of creatures (severally) and the total number of creatures (to be bought) by the price of a creature having maximum value (and then) subtract the respective prices (from the products). Choose optional multipliers such that (the sum of their) products with (the respective) remainders equals (the remainder after subtraction from the product of) total number of creatures. Subtract (the sum of) the products of (the chosen) multipliers with (the respective) prices of creatures from the multiplied total price. (The reminder) divided by the multiplier (in the remaining price) is the other) multiplier (and so on). (The multipliers for the various creatures, so obtained, when multiplied by the respective rate-creatures will give the number of creatures of the respective varieties which will be bought for the specified price; and the same multipliers when multiplied by the rate-prices of the respective creatures wil! give the prices that will be paid for the creatures of the respective varieties)."[12]

Rational : Let the rates at which a number of creatures are obtained be given. The rule gives a method for obtaining the respective numbers of creatures and their respective prices, if total number of creatures and their total price are known. let the rates for the birds be a swans for $\acute{a}$ *drammas*, b slow moving birds for $\acute{b}$ *drammas*, c parrots for $\acute{c}$ drammas and d cuckoos for $\acute{d}$ drammas and let their numbers which has been called *Multiplier* by the author, be x, y, z amd t, respectively. Also, let the total number of birds be S and their total price be P. Then, we have $ax + by + cz + dt = S$..................(1) and

$$\acute{a}x + \acute{b}y + \acute{c}z + \acute{d}t = P, \qquad ...(2)$$

Also, suppose that $\frac{\acute{d}}{d}$ is greatest amongst $\acute{a}/a$, $\acute{b}/b$, $\acute{c}/c$ and $\acute{d}/d$. Multiplying both the sides of (1) by $\acute{d}/d$ and then subtracting (2) from the product, we get,

$$\left(\frac{a\acute{d}}{d} - \acute{a}\right)x + \left(\frac{b\acute{d}}{d} - b\right)y + \left(\frac{c\acute{d}}{d} - \acute{c}\right) = \frac{S\acute{d}}{d} - P.$$

This indeterminate equation in x, y, z, is solved by one's own intelligence and substituting these values in (1) or (2), value of t is obtained. These value of x, y, z, and t when multiplied by the respective rate creatures (i.e., a, b, c and d) give the numbers of creatures of the respective varieties and the same multipliers when multiplied by the respective rate prices (i.e., $\acute{a}$, $\acute{b}$, $\acute{c}$ and $\acute{d}$) give the prices of the respectives creatures.

Ex.38." Arranged in rows for sale, 5 slow moving swans are obtained for 3 (drammas), 7 talkative parrots, for 5 (*drammas*), 9 peacocks, for 7 *drammas* and 3 cuckoos humming in feeble voice, for 9 *(drammas)*. O learned, bring 100 birds for 100 *drammas* for the pleasure of the king."

THE CISTERN PROBLEM

R.36(a). "Divide unity by the (given) fractions (of time, severally). Unity,

divided by the sum (of the above quotients), happens to be the time of filling the cistern (when all the drains are simultaneously opened to flow into it)."[13]

The rule states that if n drains severally fill up a cister in $\frac{a}{b}, \frac{a''}{b''}, \ldots\ldots, \frac{a^k}{b^k}$ of a day; all the drains working together will fill the cistern in

$$\frac{1}{\left(\frac{a}{b} + \frac{a''}{b''} + \ldots\ldots + \frac{a^k}{b^k}\right)}$$

of a day.

Ex.39. "Tell, in what part of (a day) will 4 fountains fill a cistern, working simultaneously, which they severally fill in 1, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of a day."

PURCHASE AND SALE (CONTINUED)

R.36(b)-37(a). "The greatest capital increased by an optional number is *kraya*. That (i.e., *Kraya*) is multiplied by the sum of the price at which the remnant article is sold and the multiplier. The product less 1 happens to be vikraya."[14]

Karya is the rate at which articles are purchased for 1 *rupa* and *vikraya* is the rate at which articles are sold for 1 rūpa. Let n persons whose capitals are $x,y,z,\ldots$, purchase certain articles at the rate of p articles per *rūpa* and sell these articles in multiples of s at the rate of s articles per *rūpa*. Suppose, by selling the remnant articles at the rate of r rūpas per article and then subtracting m times their respective investments from their total sale proceeds, they become equally rich. Let t be the greatest of $x,y,z,\ldots$. and k, an optional number. According to the rule,

$$p = s\,(r + m) - 1, \text{ and } s = t + k$$

Rational : Let us suppose that by selling the articles in multiples of s, the persons earn A, B, C, ..., rūpas, respectively. Then the total amounts earned by n persons are, $A + r\,(xp - As)$, $B + r\,(yp - Bs)$, $C + r\,(zp - Cs)$,...., in order. Since their total sale proceeds less m times their investments are equal, so,

$$A + r\,(xp - As) - mx = B + r\,(yp - Bs) - my = C + r\,(zp - Cs) - mz = \ldots\ldots$$

or, $$rxp - mx - A\,(rs - 1) = ryp - my - B\,(rs - 1) = rzp - mz - C\,(rs - 1) = \qquad \ldots(1)$$

To solve (1) suppose that $p = s\,(r + m) - 1$.

Substituting this value of p in (1) and simplifying, we get,

$$(rs - 1)\,[(r + m)\,x - A] = (rs - 1)\,[(r + m)\,y - B]$$

$$= (rs - 1)\,[(r + m)\,z - C)] = \ldots\ldots$$

So, (1) will hold, if $p = s\,(r + m) - 1$ and

$$x\,(r + m) - A = y\,(r + m) - B = z\,(r + m) - C = \ldots\ldots\ldots\ldots = 1$$

Let k, v, ..., be the remainders when xp, yp,.....are severally divided by s, in order. Then

$$xp/s = A + k/s\ ;\ yp/s = B + v/s, \ldots\ldots$$

Also, when $p = s\,(r + m) - 1$; $\quad A = (xp/s) - (k/s) = x\,(r + m) - (x + k)/s$

Similarly, $B = y(r+m) - (y+v)/s$ and so on. Thus s must be greater than the greatest among x, y, z,.... Therefore a solution of (1) is

$$p = s(r+m) - 1,\ s = t + k \text{ and hence the rule.}$$

Obviously, equal wealth in this case will be given by $(sr - 1)$.

"(Four merchants) purchase bananas for 6, 10, 50 and 100 *paṇas* and sold them at the same rate, selling the remnants at the rate of 1 for 6 *paṇas*. O friend, with the profit, they became equally rich or (became equally rich) after subtracting their (respective invested) capitals or (became equally rich) after subtracting twice their (respective invested) capitals. O mathematician, if (you) have not fallen from the peak of your pride, tell (the rates of) purchae and sale."

R.37(b)-38(a). "Reducing the capitals to their common denominator and removing the denominators, obtain *kraya* and *vikraya* by the method stated earlier. The common denominator multiplied by *kraya,* so obtained, happens to be *vikraya* in case of fractional capital."[15]

Let n persons invest in purchasing and selling articles and suppose that their capitals are $\acute{x}$, $\acute{y}$, $\acute{z}$, ... where $\acute{x}$, $\acute{y}$, $\acute{z}$,...... are fractions. Also, let p, s, r, t, and k have the same meanings as in R.36(b)-37 (a), but here suppose that no subtraction of capitals takes place i.e., $m = 0$. Let $\acute{x}$, $\acute{y}$, $\acute{z}$, ... reduced to a common denominator be x/D, y/D, z/D,..... to the rule, $p = D(s.r - 1)$ and $s = t + k$.

Rational : Taking x/D, y/D, z/D,.... in place of $\acute{x}$, $\acute{y}$, $\acute{z}$,.... in order, and $m = 0$, as in R.36 (b) - 37 (a), here we find that equations (1) of that rule takes the form

$$rxp - AD(rs - 1) = ryp - BD(rs - 1) = rzp - CD(rs - 1) = \ldots\ldots$$

Now supposing $p = D(rs - 1)$ and proceeding as in the previous rule, this rule can be established, Obviously equal wealth in this case also will be given by $(sr - 1)$.

Ex.41. "(4 persons) purchasing fruits of plantain tree (at the same rate) for 1/2, 1/3, 1/5 and 1/4 (of a *rūpa*) and selling them at the same rate and selling the remnants at the rate of 4 (*rūpas*) for1 (fruit) became, equally rich. O learned tell *kraya* (and) vikrya."

R.38(b). An integer exactly divisible by all the denominators is called Śuddha. And śuddha to an optional number. (The sum) multiplied by the denominator of *vikraya*. After that multiply (vikraya) by the value of vikraya of the remnant articles. (The product) less 1 happens to be the equal wealth. Multiply that (i.e., the equal wealth) by the denominator of vikraya of the remnant articles. (The product) multiplied by śuddh, certainly determines the value of kraya in case of fractional or non-fractional capitals."

In this case, let $\acute{x}$, $\acute{y}$, $\acute{z}$, ... x/D, y/D, z/D,.... n, p, s, t and k have the same meanings as in R.37(b)-38(a). Suppose that $m = 0$, and after selling the remnant articles at the rate of r/d *rupas* per article, the persons become equally rich. According to the rule, $p = dD(sr/d - 1)$, $s = d(D + k)$ and equal wealth $= (sr/d - 1)$.

Rational: Taking x/D, y/D, z/d,.... in place of x, y, z,.... and $m = 0$, as in the previous rules, equation (1) of those rules takes the form,

$$rxp - AD\, d(rs/d - 1) = ... ryp - BDd\,(rs/d - 1) = rzp - CDd\,(rs/d - 1)$$

Now supposing $p = dD\,(rs/d - 1)$ and proceeding as in those rules, the rule can be established. Obviously equal wealth in this case will be given by $(rs/d - 1)$.

Ex.42, "(Four persons) obtaining fruits of the plantain tree at the same rate for 1/2, 1/3, 1/4 and 1/5 of a paṇa sold them at an equal rate. Selling the remnant (fruits) at the rate of 5/4 *paṇas* for 1 (fruit), (they) became equally rich. What are the rates of purchase and sale."

Ex.43. "The capitals of 4 persons are 1/2, 2/3, 3/4 and 4/5. They became equally rich by purchasing and selling articles at the same rates) and by selling the remnant articles at the rate of 9/8 (for 1). Friend, what are the equal rates of purchase and, sale."

Meeting of Two Travellers.

R.39. "The distance divided by the sum of speeds happens to be the time for the first meeting. Twice (the quotient, obtained by the division of distance of distanc by the same (is the time of) meeting again after that meeting."

Let d be the distance between two places and suppose that two persons started from these places simulatneously in opposite directions with speed s_1 and s_2. If t_1 and t_2 be the times of their first and second meetings,

according to the rule, $t_1 = d/(s_1+s_2)$ and $t_2 = 2d\,(s_1+s_2)$.

Rational. Since the distance between the points of start of two travellers is d and the sum of their speeds is $s_1 + s_2$ and also they move in opposite directions, clearly, $t_1 = d/(s_1 + s_2)$.

Now the distance covered by the two travellers before the second meeting = $t_2s_1 + t_2s_2$ and that before the first meeting is d.

So, $(t_2s_1 + t_2s_2) - d = d$

or $t_2 = 2d/(s_1 + s_2)$ and hence the rule.

Ex.44. "The distance between any two towns is 300 yojanas. Two letter-carriers started from their respective towns (simultaneously), one with a speed of 11 yojanas and the other, 9 *yojanas*, per day. O learned, if (you) know, tell quickly the times of their two meetings, (the first) after their start and (the second) while returning back."

Ex.45. "One letter-carrier moves with a uniform speed of 9 *yojanas* and the other, 5 yojanas, per day. The distance (between them) is 210 yojanas. O friend, tell the times of their meeting in days."

R.40 (a). "Length of the circumference, divided by the difference of speeds,

happens to be the time of meeting."

Let S and s be the speed of two persons moving on the circumference (= C) of a circle in the same direction starting simultaneously at the same place. If $S > s$, according to the rule, the time of their meeting = $c/(S - s)$.

The rule is obvious as the relative velocity of the travellers is $S - s$.

Ex.46. "One (traveller) moves with a speed of 8 *yojanas* and the other, 5 *yojanas* (per day). The circumference of the country on which they move is 120 yojanas. Tell, after how many days they will meet together at their town if both the travellers (start simultaneously) and move in their path in the same direction."

PORTER'S WAGES PAID FROM THE COMMODITY

R.40(b). "In case of (a part of the merchandise) falling down, the wages for half the quantity fallen down together with (full wages) for the rest happens to be the total wages."

Let p be the wages prescribed for carrying oil whose weight is W. Also, suppose that oil whose weight is w trickles down in the way. According to the rule the wages to be paid to the porter will be $(p/W)\ [w/2 + (W - w)]$.

The wages prescribed in this rule correspond to the average load carried by the porter.

Ex. 47. "A porter obtains 9 *paṇas* for carrying 300 palas of oil in a leathern oil pot to a distance of 10 yojanas. Some oil trickles down (in the way) through hole in the leathern oil pot. The rest is 60 *palas*. Tell the wages to be paid to the porter."

R.41-42(b). "Multiply the stipulated distance by the merchandise. (The product) divided by the distance (which remains) to be gone is 'the sum'. Subtract the merchandise from the sum. (The difference) multiplied by the measure of wages is 'the product". (Find) the difference from 'the sum' and 'the product'. (Obtain the roots) from them by (the method of) concurrence. The smaller root is the wages (to be paid to the porter)."[16]

Let the stipulated wages for carrying a articles over distance d be w articles. Also, suppose that the wages for carrying a articles over a distance s, where $s < d$, is x articles. According to the rule, $[ad/(d - s) - a]w$ = the product and $ad/(d - s)$ = the sum of the roots. The roots are to be obtained by R.35 and R.31of chapter I. The smaller root will be the wages to be paid to the porter.

Rational : Calculating at the same rate, the wages for carrying the remaining $(a - x)$ articles over the remaining distance $(d - s)$ are equal to $(a - x)\ (d - s)\ x/as$ articles. But the remaining wages are $(w - x)$ articles. So, $(a - x)\ (d - s)\ x/as = w - x$

or $\quad (d - s)x^2 - adx + aws = 0$ and hence the rule.

The greater root has to be rejected as this will make $x > w$.

Ex.48. "The wages for carrying 100 oranges for 12 krośas is 37 oranges. O eminent mathematician, praised in the assembly of mathematicians, tell quickly the wages to be given if the distance covered is (only) 3 krośas."

True and False statements

R.42 (b)-43 (a). "Add 1 to (the number of) persons liked (by a lady). Multiply (the sum) by the number of persons. (The product) less twice (the number of) persons liked is the number of false statements. Subtract the same from the square of the number of persons. The remainder is the number of true statements."[17]

Let a lady like n persons out of m persons while others are disliked by her. Suppose that she speaks to each person that only he is liked, while others, disliked. Then, the number of false statements $= m(n+1) - 2n$, and the number of true statements $= m^2 - m(n+1) + 2n$.

Rational : Here, the total number of statements made by the lady $= m^2$. Also, n out of m persons are liked by her and so, $(m - n)$ person are not liked by her. When she says to each of the n persons liked by her, that only he is liked by her, the number of false statements in each case is $(n - 1)$ and so, the number of false statements in such cases = n(n-1). Again when she says to each of $(m - n)$ persons not liked by her, that only he is liked by her, the number of false statements in such cases $= (m - n)(n + 1)$. So, adding these two, total number of her false statements $= m(n + 1) - 2n$. Also, total number of statements is m^2. Therefore, total number of her true statements

$$= m^2 - m(n + 1) + 2n$$

Ex.49. "2 person, out of 54, in love with another's wife, are liked by her and the others, disliked. She talks in wonderful language to each of them, separately, that only he is liked by her (and the others, disliked). Tell, quickly the number of (her) true and false statements (separately)."

REFERENCES

1. Cf. *BSS,* chap. xii, 16 (a); *PG*, 59(i); *Tri.* vs. 38(1); *SŚ*, 29; *L*, 52.
2. Cf. *PG*, 59(ii); *Tri*, vs. 38 (ii); *MS*, vs. 37-38 (a) *SŚ*, 29; *L*, 54.
3. See also, *GSS*, 98-99.
4. Cf. *PG*, 32-33.
5. Cf. *GSS*, 135-37; *PG*, 36-38, *Tri.*, vss. 36-37, *MS*, vss. 39(b)-40; *L*, 58-59.
6. Cf. *PG*, 38; *MS*, vs., 39; *L*, 59.
7. Cf. *PG*, 38-39; *MS*, vs. 41(a); *L*, 60.
8. Cf. *L*, 56.
9. Cf. *GSS*, 133-34.
10. Cf. *GSS*, 153-54.
11. See also, *GSS*, 155-56.
12. Cf. *PG*, 50-52. See also, *GSS*, 129-30.

13. Cf. *PG*, 55-56; *MS*, vs. 43;*L*, 53-54.

14-15. Cf. *PG*, 44-45.

16. See also, PG, 56-57.

17. Cf. *GSS*, 145-46.

CHAPTER III

In this chapter, the following notations are used.

Symbol	*Stands for.*
a	First term of a progression.
l	Last term of a progression.
n	Number of terms of a progression.
d	Commo difference of an arithemtic progression (=A.P.).
r	Common ratio of a geometric progression (=G.P.).
s	Sum of a series.
S	Sum of first n natural numbers.
$C(n, r)$	Number of combinations of n different things taking r at a time.

In this chapter A.P. is the first topic of treatment (R. 1-6). After that 'meeting of travellers' has been dealt with [R.7-9(a)]. Next, the sum , the sum of squares and the sum of cubes, of first n natural numbers find place [R.9(b)-19(a)]. 'Rth sum of a natural number' and' the cow problem' is the next topic of treatment [R.19(b)-22]. Lastly G.P. has been dealth with (R.23-28).

DETERMINATION PERTAINING TO PROGRESSION

ARITHMETIC PROGRESSION (=A.P.)

R.1-2(a)."Multiply 'the number of terms less1' by the common difference. (The product) added to the first term is the last term. The last term added to the first term and then halved is the average (of all the terms). The average multiplied by the number of terms happens to be the sum (of the series in A.P.). Multiply half of 'the number of terms less 1' by the common difference. Add the product to the first term. (The sum,) multiplied by the number of terms happens to be the sum (of the series in A.P.)."[1]

According to the rule, (i) $l = a + (n - 1)d$; (ii) $m = (a + l)/2$,

(iii) $s = mn$, and also, (iv) $s = n[a + (n - 1)d/2]$.

Series in A.P. are found to occur in the *Taittiriya Samhitā*, the *Vājasaneya Samhita*, the *Pancavimsa Brahmaṇa* and other vedic works and in the *Bṛhaddevatā* (500-400 B.C.). Rules for finding the sum etc. of a series in A.P. occur in the *Bakshali Manuscript* and other early Indian works.

Ex.1. "A rich man gives a senior Brāhmaṇa 3 *niṣkas* on the first day and the same amounts increased by 2 *niṣkas* on (each) succeeding day, for half a month. Friend, if you know mathematics, tell the last (day's) amount, the average and the total amount."

Ex.2. "A trader earns 2 on the first day and 3 more on each succeeding day, for 6 days. Tell quickly the average, the last term and the total amount."

R.2(b). "Divide the sum by the number of terms. (Also) multiply 'the number of terms less 1' by half of the common difference. Subtract (the product from the earlier quotient. The result is) the first term."[2]

According to the rule, $a = (s/n) - (n - 1)d/2$.

Ex.3. "If you have sufficient interest in progression, tell the first days's travel of a person travelling at an increasing rate of 3 *yojanas* per day if he travels 168 (*yojanas*) in 7 days."

Ex.4. "The first term is 4, the number of terms, 9 and the sum, 216. O learned, tell me the common difference."

FINDING THE NUMBER OF TERMS

R.3. "Multiply twice the common difference by the sum. Subtract half of the common difference from the first term. Add the square of the difference to the product. Subtract the first term from the square-root of (the resulting) sum (and then) add half of the common difference (to that). The total divided by the common difference (is) the number of terms."[3]

According to the rule,

$$n = \left[\sqrt{2ds + (a - d/2)^2} - a + d/2\right] / d$$

From R.2(a) we get a quadratic equation in n. The product of the roots of this equation being negative, one root will be positive. The rule gives this value of n.

Ex.5. "The first term is 8 and the common difference, 6. Friend, tell me quickly the number of terms by which the sum (of the series in A.P.) will be 288."

R.4. "Desired number of terms and the first term or the desired common difference should be obtained from the knowns and (supposing the rest unknowns) by their own rules stated earlier."

Ex.6. "O learned, if you are proficient in the line of series, tell the forgotten first term etc. where the sum of the series in A.P. is 100."

R.5. "The product (of the first term, the common difference and the number of terms) divided by the desired first term is called *Krama*. Add the *krama* to twice

the first term. Multiply the sum by half of the *krama* added to the sum (of the series. The quotient) is the common difference. Here, the *krama* divided by the common difference is the number of terms."

According to the rule, if k be the *krama*.

(i) $k\ (= and/a) = nd$;

(ii) $d = (2a + k).\ (k/2)\ /\ (s+k/2)$; (iii) $n = k/d$.

Ex. 7. "The product of the first term, the number of terms and the common difference is 12 and the sum of the series is 10. O friend, tell them (separately). What are the measures of the first term, the number of term and the common difference, if their product and the sum (of the series) be equal to 1 (each)."

R.6. "One (progression) has greater first term and lesser common difference; the other has (lesser) first term and greater common difference, (the sum of the series and, their number of terms being equal). Divide the difference of their first terms by half of the difference of their common differences. (The quotient) added to 1 is the number of terms."

Let a be the first term and d, the common difference of the first progression, A,D being those of the other, respectively; the number of terms ($=n$) and sum of terms being equal.

Let $a > A$ and $d < D$. According to the rule,

$$n = (a - A)/(1/2)\ (D - d) + 1$$

Ex.8. "A traveller travels, 9 *yojanas* on the first day and 5 *yojanas* more on each subsequent day. The other travels 2 *yojanas* on the first day and 7 *yojanas* on each subsequent day. Tell how many days and how much they travel (if they travel equally)."

Meeting of two travellers

R.7-8. "The first term and the 'not lesser' common difference of the first (progression) and the number of terms of the second progression are desired. Supposing the first term (etc.) of the first (progression), obtain the sum (of the series in A.P.). Obtain the first term of the second, from the common sum. The first term added to the product of 'the number of terms and the common difference' happens to be the first term for he second meeting (for each traveller. From those, the sum (of the series in A.P.) and the number of days (i.e., terms) should be obtained by the method stated earlier."

Let a and n be the initial speed and the number of days for the first traveller and A and N be those for the second traveller for the first meeting and $\acute{a}$, $\acute{n}$ and $\acute{A}$, $\acute{N}$ refer to those, in order, for the second meeting. Suppose that the first traveller starts with an increasing speed d per day and the second, that with D per day. Also let the second traveller start when the first traveller has move for t days and, $d \geq D$. D and t are given. According to the rule, a, d and N are supposed. Now, $n = t + N$ and,

s and A are obtained by the methods stated earlier and considering the fact that the distance traversed by the two travellers for the first meeting is the same.

For the second meeting, according to the rule, $\acute{a} = a + nd$, and $\acute{A} = A + ND$. Clearly $\acute{a}$ and $\acute{A}$ are the terms next to the last term for the first meeting. Rule 6 gives the time $\left[= \frac{2(\acute{A} - \acute{a})}{(d-D)} + 1 \right]$ between two meetings and so, $\acute{N} = \frac{2(\acute{A} - \acute{a})}{(d-D)} + 1 + N$, and $\acute{n} = t + \acute{N}$ and therefore the distance covered before the second meeting can be found out.

According to Nārāyaṇa, in case the time obtained between two meetings is negative ($= -t$), $\acute{N}$ ($= N - t$) and $\acute{n}$ ($= n - t$) give the number of days for the first meeting and N and n, those for the second meeting.

Ex.9. "Someone goes to a town with a certain initial speed per day with an increasing rate. Other person follows him after 8 days with certain (other) initial speed with an increasing rate of 2 per day. The two friends met twice in the way. O learned in the science of progression, like a king of lion in its fight against a king of elephant, if you know, tell the number of days (etc., when they meet each other)."

R.9 (a). "Subtract the first term from the constant speed. Divide (the difference) by half of the common difference. 1 added to (the quotient) is the number of terms."[4]

Ex. 10. "Remembering her husband eagerly, a lady goes with a constant speed of 10 yojanas per day. A procuress follows her with an initial speed of 3 (yojanas per day), increasing with 2 (yojanas per day, each day). After how many days, they meet?"

SUM OF FIRST N NATURAL NUMBERS, ALSO OF THEIR SQUARES, CUBES, ETC.

R. 9 (b)-10. "For the sum of first n natural numbers and the square of a number, take 1 as the first term and 1 and 2 (respectively) as the common differences. For the cube of a number, take the number as the first term and twice of that as the common difference. For the sum of any two or more of the above, take the sum of (the respective) first terms as the first term and that of the common differences as the common difference."

Ex.11. "Friend, tell me the sum of first n natural numbers, the square of 5 and the cube of 5, separately and the sum of any two or all of these by the method of (summation of a series in) A.P."

R.11. "And 1 to the number of terms ($= n$, say). (The sum) multiplied by half of the number of terms, is the sum of first n natural numbers. (The product) is called *Aya*. Thrice the āya, less the number of terms, is equal to the sum of the square (of n) and the sum of first n natural numbers."

R. 12. "And the sum of first n natural numbers to the square of n. (Also), add

6 times the number of terms to 1. Multiply (the two sums). Add 4 times the number of terms to (the product). (The result) divided by 9 is the sum of the cube of n added to) the sum of first *n* natural numbers."

R.13 (a). "Twice the sum of first n natural numbers multiplied by the number of terms happens to be the sum of the square and the cube of *n*."

R.13.(b). "Add twice the number of terms to 1. Multiply the sum by the sum of first *n* natural numbers. (The product) is the sum of the sum of first *n* natural numbers, the square of *n (and* its) cube."

R.14 (a)-15. "Add 2 to the number of terms. Multiply the sum by the sum of first n natural numbers. The product divided by 3 is the sum of the sums of first *n* natural numbers up to *n*. Add (together) the sum of the first *n* natural numbers, the square and the cube of *n*. The sum divided by 3 is the sum of the squares of first *n* natural numbers. Square of the sum of first *n* natural numbers happen to be the sum of cubes of the first n natural numbers."[5]

The rule states that

(i) $\sum_{r=1}^{n} \frac{r(r+1)}{2} = \frac{n(n+1)(n+2)}{2.3}$

(ii) $1^2 + 2^2 + ... + n^2 = (S + n^2 + n^3)/3$

(iii) $1^3+2^3+.....+n^3 = S^2$, where as usual, S is the sum of first n natural numbers i.e., $S = n(n+1)/2$.

Ex.12. "O learned, if you know the way to progression tell the sum of the sums of, the squares of and, the cubes of natural numbers from 1 upto 5."

SERIES OF ĀYAS (I.E., SUM OF FIRST N NATURAL NUMBERS), SQUARES AND CUBES

R.16-17(a). "Subtract the sum of first natural numbers upto the first term (of a progression having *n* terms) from the sum of first natural numbers upto the sum of first term and the common difference (of the progression). Multiply the difference by the sum of first (*n* - 1) natural numbers. (Also), multiply the sum of first natural numbers upto the first term (of the progression) by (its) number of terms. (Again), multiply the square of the common difference by the sum of the sums of first natural numbers upto (*n* - 2). Add (the product to the sum of above two products). The sum is the sum of sums of first natural numbers upto (each) term (of the progression)."[6]

If a progression be,

$a, a + d, a + 2d, ..., a + (n - 1)d$, then according to the rule, $(1 + 2 + 3 + ... + a) + (1+2+....+a+d) + (1+2 +...+ a + 2d)+....+(1+2+...+ a + \overline{n-1}\,d)$. i.e.,

$$\sum_{r=1}^{n} \frac{[a+(r-1)d][a+(r-1)d+1]}{2}$$

$$= \frac{n(n-1)}{2}\left[\frac{(a+d)(a+d+1)}{2} - \frac{a(a+1)}{2}\right] + \frac{na(a+1)}{2} + \frac{n(n-1)(n-2)}{1.2.3}\, d^2$$

Ex.13. "O friend, if you are an expert in mathematics, tell the sum of (a series of) sums of first natural numbers, that of (a series of) squares and the same of (a series of) cubes, the first term, the common difference and the number of terms of the progression being 3,4 (and) 9, (respectively)."

R.17(b)-18(a). "Multiply 'the sum of a series (in A.P., obtained with) the common difference increased to its double' by the first term. (Also), multiply 'the sum of squares (of first natural numbers whose) number of terms is 1 less (than the number of terms of the A.P.). The sum (of the above products) is the sum (of the series) of squares (corresponding to the terms of the A.P.)."[7]

According to the rule,

$$\sum_{r=1}^{n}[a+(r-1)d]^2 = a\left[\frac{n}{2}(2a+\overline{2n-2}d)\right] + d^2\left[1^2+2^2+\ldots+(n-1)^2\right]$$

R.18(b)-19.(a). "Multiply the square of the sum of the series (in A.P.) by its common difference (Also), multiply the difference of the first term and the common difference (of the A.P.) by its first term and them by the sum (of the series in A.P.). Add (the latter product to the former product if) the first term (of the A.P.) is greater than its common difference (and) subtract, if otherwise. (the result) is the sum (of the series) of cubes (corresponding to the terms of the A.P.)."[8]

According to the rule,

$$\sum_{r=1}^{n}[a+(r-1)d]^3 = s^2 d + a(a-d)\, s,$$

where s is the sum of the series a, $a + (a + d) + (a+2d) + \ldots + [a+(n-1)d]$

*R*TH SUM OF A NATURAL NUMBER

R.19(b)-20(a). "Number of terms (say n, is) the first term (of an A.P.) and 1, the common difference. Those (i.e., terms of the A.P., their number being) 1 more than the number of times (the sum is to be taken, i.e., $r + 1$), separately, (are) the numerators. 1 (is) the first term (of another A.P.) and 1, the common difference. (These are the denominators, (their number being the same as that of the former A.P.). Their product (is) the rth sum of n."

Here the rth sum of n

$= \Sigma\Sigma\ldots\ldots\Sigma n$,

the number of Σ s being r.

According to the rule, *rth* sum of n

$= n\ (n + 1)\ (n + 2)\ldots(n + r)/1.2.3\ldots.(r + 1)$

Clearly, *rth* sum of $n = C\ (n + r, r + 1)$.

Ex. 14. "O learned friend, if you have passion for mathematics, tell the 3rd sum of 6, the first term being 1 and the common difference, 1."

RTH SUM OF A SERIES IN A.P.

R.20(b)-21. "The *rth* sum of 'number of terms less 1' is the multiplier (of the common difference). That separately multiplied by '1 more than the number of times' (the sum is to be taken, i.e, $r + 1$ and then) divided by 'the number of terms less 1' is multiplier of the first term. The first term and the common difference (are both) multiplied by their own multipliers. The sum of the products happens to be the rth sum (of the series in A.P.)."

Let an A.P. be, $a.\ a + d,\ a + 2d,\ldots,\ a + (n - 1)d$.

According to the rule, *rth* sum of the series in A.P.

$$= \left[\frac{a(r+1)}{(n-1)} + d\right] \frac{(n-1)\,n\,(n+1)\ldots(n+r-1)}{1.2.3\ldots\ldots(r+1)}$$

Rational : First sum of the series in A.P. is $\left[an + \frac{n(n-1)}{2} d\right]$

So, *rth* sum of the series = $(r - 1)$th sum of $\left[an + \frac{n(n-1)}{2} d\right]$

$$= a.\ [(r - 1)\text{th sum of } n] + d\ [\ rth \text{ sum of } (n - 1)]$$

$$\left[\text{as } \frac{n(n-1)}{2} \text{ is the first sum of } (n-1)\right]$$

which leads to the desired result.

Ex.15. "First term (of an A.P.) in 5, common difference, 3 (and) the number of terms, 7. O best among learneds, tell quickly the 4th sum (of the series in A.P.). (Also), if you have passion for mathematics, tell the sum by changing (the ingredients)."

COW PROBLEM

R.22. "Subtract the number of years (in which a calf begins giving birth) from the (total) number of years (successively and) separately, till the remainder becomes smaller (than the subtractive). Those are the numbers for repeated summations once, (twice) etc. in order. Sum of the summations along with 1 added to the total number of years is the number of progeny."

Ex. 16. "A cow gives birth to a (she) calf every year (and) their calves themselves begin giving birth) when 3 years old. O learned, tell the number of

progency produced by a cow during 20 years."

Solution : Successive subtractive of 3 from 20 gives, 17, 14, 11, etc. So, total number of progeny = 1st sum of 17 + 2nd sum of 14 +3rd sum of 11 + ...+ 6th sum of 2 +1+20

$$(= 153 + 560 + ... + 8 + 1 + 20) = 2745.$$

According to Nārāyaṇa, '*rth* sum of *n*' is a characteristic of *chanda* (metre) and so can be obtained by the method stated in *ankapāśa*. It can also be obtained from *meru*. These method have been stated later on.

GEOMETRIC PROGRESSION (=G.P.)

R.23. "If the number of terms be odd, subtract 1 (from it and write) 'multiply (by the common ratio and if) even, halve (the number of terms and write) 'square'. (Continue the process till zero is obtained). Starting from the last (i.e., 1 and moving backwards, (perform the operation of) multiplication and squaring (as set down). Subtract 1 (from the result). Divide (the difference) by 'the common' ratio less1'. (The quotient) multiplied by the first first term is the sum (of the series in G.P.)."[9]

According to the rule, $s = a(r^n - 1) / (r - 1)$. Shukla, has given details of the process. 10 Ex.18 has been solved by the process (in short).

Ex.17. "A certain rich man gave a Brāhmaṇa 2 *varātakas* on the first day and on each subsequent day, (the previous days amount) multiplied by 3. O eminent mathematician, if you have passion for mathematics, tell quickly the total amount (given) in a month."

Ex.18. "Tell the sum (of the series in G.P.) in which the first term is 3, the common ration, 2 and the number of terms, 7. (Tell) the first term and the number of term from that sum."

Process. Here, $a = 3$, $r = 2$, and $n = 7$.

Number of terms	*Operation*	*write*
7 (odd)	7-1 (=6)	multiply
6 (even)	6/2 (=3)	square
3 (odd)	3-1 (=2)	multiply
2 (even)	2/2 (=1)	square
1 (odd)	1-1 (=0)	multiply

Process from the last (i.e., 1) backwards and performing the operations of 'multiplication' by the common ratio 2 and 'squaring', successively, as indicated in the last column, we get, 2, 2^2, 2^3, 2^6 and finally 2^7, and therefore, $s = 3(2^7-1)/(2-1) = 381$.

FINDING THE FIRST TERM

R.24. "Multiply the sum by 'the common ratio less1'. The product divided by 'the result of multiplication and squaring' less 1 happens to be the first term."

FINDING THE COMMON RATIO

R.25. "Divide the sum by the first term. Subtract 1 (from the quoteint). Divide (the difference) by a factor which leaves no remainder. 'Subtract 1 from the result (and then) divide by the same factor', again and again, until 1 is obtained. The divisor is the multiplier (i.e.,) the common ratio)."

According to the rule, one has to find a factor r such that, $\frac{1}{r}(\frac{1}{r}(\ldots\frac{1}{r}(\frac{1}{r}(\frac{s}{a}-1)-1)-1\ldots1)=1$ then r will be the desired common ratio of the geometric progression.

FINDING THE NUMBER OF TERMS.

R.26-27(a). "Divide the sum by the first term. Multiply (the quotient) by 'the common ratio less 1'. Add 1 (to the product). Divide (the result) by the multiplier (i.e., by the common ratio) again and again till the end. (for each division) keep unity. The sum of that (i.e., of unities) is the number of terms."

According to the rule, $(r-1)(s/a)+1$ is divided by r, again and again, till the end. Number of division gives the number of terms.

FINDING (THE NUMBER OF) <u>SAMA</u> (ETC.) VṚTTAS

R.27(b)-28. "Taking the number of letters in a fourth part of a metre as the number of terms and 2 as the common ratio, the produce of multiplication and squaring is the number (of variations) of *sama vṛtta* (i.e., of like metres). Its square and, square's square less their (respective) roots are (the variations of) *ardha sama vṛtta* and, *viṣama vṛttas* (i.e., alternately similar and dissimilar metres, respectively)."[11]

The rule is meant for *varṇa vṛttas* in which the number of letters is constant but it can either be a *laghu* or a *guru*. Let there be 4n letters in the metre (i.e., n letters in a 4th part of it). In a sama vṛtta, all the 4 parts of the metre have the same arrangement of letters. So, total number of variations of such a metre $= 2^n$.

In an *ardhasama vṛtta* half of the metre have the same arrangement of letters as the other half and so the total number of variations of such a metre $= 2^{2n} - 2^n$, as 2^n out of 2^{2n} will be the variations of *sama vṛttas*.

In a viṣama vṛtta, all the 4 parts are different, so total number of variations of such a metre $=2^{4n} - (2^{2n} - 2^n) - 2^n = 2^{4n} - 2^{2n}$

as $2^{2n} - 2^n$ and 2^n out of 2^{4n} will be the variations of ardhasama and sama vṛttas, respectively.

Ex.19. "O mathematician, if you know, tell quickly, the number (of variations) of like, alternately similar and dissimilar *vṛttas* in *anuṣṭubh chanda*."

REFERENCES.

1. Cf.*A*,61-62,*BSS*, chap. xii,vs. 17, *GSS*, 20-21; *PG*, 68;*MS* 47.

2. Cf. *GSS*, 23-24; *PG*, 69-70;*Tri.*, vs-40 (a); *MS*, vs.48; *SŚ*,35; *L*, 69-70.

3. Cf. *MS*,vs.50; *L*, 70-71. See also, *A*, 63-64; *BSS*, chap. xii, vs. 18; *PG*, 70-71; *Tri*.vs. 41;*SŚ*, 36-37.

4. Cf. *PG*, 78.

5. Cf. *A*, 64-65; *BSS*, chap. xii, vss., 19-20; *GSS*, 168, *SŚ*, 33-34; *L*, 66-67.

6. Cf. *PG*, 84-85.

7. Cf. *PG*, 84.

8. Cf. *PG*, 85-86.

9. Cf. *GSS*, 29; *MS*, vs. 52-53; *PG*, 75-77; l, 71-72.

10. Cf. *PG*. 75-77.

11. Cf. *L*, 73.

MATHEMATICIANS AND HISTORIANS

"Mathematicians are prone to identifying isomorphic structures, however faintly recognisable. Historians and philologists stress the singularity of events and texts. Both viewpoints are right in their own ways and should tolerate each other."

—B. Artmann (Mr, 96f, p.3238)

"One striking transformzation typically experienced by students encountering the history of science for the first time is that they realize that the development of science has involved not only scientific creations, but also scientific creators, not only settled conclusions, but also intense controversies. This transformation helps students to see science as an intensely human enterprise."

- Michael J. Crowe

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THE GAṆITA KAUMUDĪ OF NĀRĀYAṆA PAṆḌITA

CHAPTER IV

(English Translation With Notes)

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TRANSLATOR'S INTRODUCTION [1]

The basis and inspiration for the whole of Indian mathematics is geometry[2]. Indian geometry originated in a very remote age in connection with the construction of the altars for the Vedic sacrifices of various kinds[3]. According to strict injunctions of the Hindu scriptures, each sacrifice must be made in an altar of prescribed shape and size[4]. Thus arose in ancient India, problems in geometry. Though codified later on, the Śulbasūtras such as the **Baudhāyana Śulba Sūtra** (about 800 B.C. or later), the **Āpastamba Śulba sūtra,** the **Kātyāyana Śulba sūtra** and the **Mānava Śulba Sūtra** etc. are the oldest books on geometry which represent the much older and traditional Indian geometry[5]. Geometry of this level could not have been created overnight[6]. Some of the altars whose constructions and transformations are dealt with in **Śulba Sūtras** are mentioned in earlier treatises such as **the Taittirīya Saṁhitā** (V, 4.11) and **the Śatapatha Brāhmaṇa**[7]. More or less overlapping this period we have a voluminous Jain literature which contains a considerable amount of mathematical information including that on geometry[8].

Later on we have books on astronomy and mathematics in which a chapter or more is devoted to geometry. The former class of books such as **Āryabhaṭīya** (499 AD) of Āryabhaṭa I etc. contain a little of geometry whereas the latter class of books such as **Triśatikā of Śrīdharācārya (about 750 AD), Gaṇita-sāra-saṅgrapha** of Mahāvīra (850 AD). **Līlāvatī** of Bhāskara II (c. 1150) and **Gaṇita Kaumudī** of Nārāyaṇa Paṇḍita (c. 1356) give more or less a comprehensive account of geometrical knowlege prevalent in those times. Chapters IV to IX of **Gaṇita Kaumudī** are devoted to geometry. Chapter IV is translated now. Sanskrit text of the rules and examples follow the edition of P. Dvivedi, Part II, pp. 2-192 (Varanasi, 1942).

The following abbreviations are used.

Ā = **Āryabhaṭīya** of Āryabhaṭa I, edited and translated by K.S. Shukla and K.V. Sarma, INSA, New Delhi, 1976.

Bi = Bhāskarācārya's **Bījagaṇita** and its Translation by S.K. Abhyankara, Bhaskaracharya Pratishthan, Pune, 1980.

BSS = **Bhāhmasphuṭa-siddhānta** of Brahmagupta, edited with S. Dvivedi's commentary, Vāsanā, and Hindi translation by R.S. Sharma *et al*, 4 Vols., New Delhi, 1966.

Ex. = Example (**udāharaṇa** or **uddeśaka**)

[1] Indian Society for History of Mathematics

GB = **Gaṇita - Bhāratī.**

GK = **Gaṇita Kaumudī** of Nārāyaṇa Paṇḍita, edited by Padmākara, (Dvivedi, Benares, Part I, 1935, Part II, 1942.

GSS = **Gaṇitasāra-saṅgraha** of Mahāvīra ed. with Hindi translation by L.C. Jain, Sholapur, 1963.

GT= **Gaṇita-tilaka** of Śrīpati translated by K.N. Sinha, **GB,** 4 (1982), 112-133.

HHM = **History of Hindu Mathematics** by B.B. Datta and A.N. Singh, Lahore, Part I, 1935; Part II, 1938.

IJHS = **Indian Journal of History of Science.**

L = **Līlāvatī** of Bhāskara II, translated by H.T. Colebrooke, Kitab Mahal, Allahabad, 1967.

ME = **The Mathematics Educations** (Siwan).

MS = **Mahāsiddhānta** of Āryabhaṭa II, ed. by S.Dvivedi, Benares, 1910.

PG = Pāṭīgaṇita of Śrīdhara, ed. with an ancient commentary and transl. by K.S. Shukla, Lucknow, 1959.

R = Rule (**Sūtra**) especially from the **GK**.

SŚ = **Siddhānta-śekhara** of Śrīpati, ed. by Babuaji Misra, Part II, Calcutta, 1947.

Tri = **Triśatikā** of Śrīdhara, ed. by S. Dvivedi, Benares, 1899.

vs.vss. = verse, verses.

GAṆITA KAUMUDĪ

Chapter IV

In chapter IV, first of all, certain primary figures such as triangles and quadrangles etc. have been explained (R. 1-3a). After that rules of arithmetic and algebra necessary for further calculations have been given (R. 3b-8a) and (R. 28-31a). Next, rules giving gross results have been dealt with (R. 8b-23). Then, rules giving exact results are given (R. 24 onwards). Relation between sides and diagonals of a square and a rectangle (and a right-angled triangle) is the next topic of treatment (R. 24-27) and (R. 31b-33a). After that rules for finding areas of certain figures such as triangles, quadrangles and circles etc. have been given (R. 33b-35). Next, properties of triangles and quadrangles have been dealt with (R. 36-54a) and also (R. 111 and R. 117). Then, rules pertaing to **Sūcī Kṣetra** are given., in R. 54b-64a and those pertaing to **Śreḍhī Kṣetra** (i.e., series-figures) in R.64b-65a and R. 73-77. After that, rules regarding circles and relations between its elements such as arcs, chords etc. have been given (r. 65b-72).

Next, formation of rational and integral triangles [R.78-85; R.97b-98a; R. 118-120; R.122-123 and R.130 (b-c)] and their pairs (R. 114-115); and also formation of such quadrilaterals (R. 86-97a; R98b-101; R. 108-110; R.116 and R.124-130a) and their pairs (R.102-107 and R.112-113) find place. Towards the

end, cyclic quadrilateral, its circum-diameter and the third diagonal (R.131-148) have been dealt with. R.121 is an introductory remark. R.132 is not a rule. It is an example. R.137 is a repetition of R.131.

Ex. 22 is not an example. It is a general statement. Two different rules have been given the same serial number R.142. The chapter ends with a concluding remark.

ENUMERATION OF PRIMARY PLANE FIGURES

R1-3a "Triangles are of 3 types : equilateral, isosceles and scalene. Quadrangles are of 5 types ; square, rectangular, equi-bilateral, equi-trilateral and inequilateral. These are the eight separate regular figures. Circular figures are of 2 types : a circle and a conch figure. (Figures like) the cresent moon, the arc the annual figure, (figure resembling) the tooth of an elephant and the thunderbolt etc. are fragments of a circle."

Mahāvīra enumerates 8 varieties of curvilinear figures[9]. His classfication of quadrilaterals differs from that of Śrīdhara[10].

OPERATIONS OF ARITHMETIC

R3b-4 "Mathematicians say that calculations regarding land are of 2 types: gross and exact. Gross calculation is for practical use and exact calculation is for finding accurate result. A mathematician who knows both type of calculations is the prime among mathematicians. May he live long."

ADDITION

R. 5. "In the addition of 2 positive or 2 negative numbers (the result is their) sum, (In the case of) a positive and a negative number, subtracting the smaller from the greater, (the sum is their difference, the remainder being of the same denomination (as the greater)."[11].

SUBTRACTION AND MULTIPLICATION

R. 6 " Of the subtrahend, affirmation becomes negation and negation, affirmation. Then, add as described above[12].

The product of 2 positive or 2 negative numbers is positive. The product of a positive number and a negative, is negative[13]".

DIVISION

R. 7a . "The characteristic in the case of multiplication of positive and negative numbers also hold in the case of division[14]".

SQUARE AND SQUARE-ROOT

R.7b-8a "The square of a positive (or) a negative number is positive. The square-root of a positive number is (either) positive or that (i.e., negative) also. A negative number, being a non-square, is thus proved to have no square-root[15]."

GROSS AREA OF A TRIANGLE OR A QUADRILATERAL

R.8b. "In a triangle or a quadrilateral the product of halves of the sums of opposite sides is the gross-area[16]."

In a triangle, the face (i.e., the side opposite to the base) is considered to be a cipher.

If a, b, c and d, be the sides of a quadrilateral, in order, according to the rule, its gross area = $\left(\frac{a+c}{2}\right)\cdot\left(\frac{b+d}{2}\right)$.

This 5000-year old formula is variously called ' Surveyor's Formula or 'Taxman's Rule' etc[17]. This most popular ancient formula was universally used in almost all cultures of antiquity[18]. The formula is notewrorthy in its use for practical mensuration continuously from the ancient through the medieval, down to modern times.[19]

Ex.1 "O Mathematician, tell me the area of a square whose lenght is 20 **daṇḍas** and (that) of a rectangle whose lenght is 24 and breadth, 18 **(daṇḍas)**".

Ex. 2-3 " If you are skilled in the rules of mathematics, tell the gross area of a equi-bilateral (quadrilateral, whose) flank sides are 5 (each), face is 2 and base, 8 and (that of) a equitrilateral (quadrilateral, whose) flank sides and (base) are 5 (each) and, face is 11 (and also that of an inequilateral quadrilateral whose) flank sides are 5 & 7, face is 4 and base, 10."

Ex.4 "What is the area of an equilateral triangle (whose) sides are 12 (each and that of) an isosceles (triangle whose) flank sides are 10 (each) and, base is 12 (each also that of) a scalene (triangle whose) flank sides are 21 and 17 and, base is10."

GROSS CIRCUMFERENCE OR AREA OF A CIRCLE

R.9 "In a circle, thrice the diameter is the circumference (and the circumference) multiplied by one-fourth of the diameter is the area. Thrice the square of the diameter divide by 4 or (thrice the square of) the circumference divided by 36 is[20](also) the area."

Let C be the circumference, d, the diameter and A, the area of a circle, the constant ratio C/D is now universally denoted by π. According to the rule, for gross calculations (taking $\pi = 3$), $C = 3D$; $A = C.D/4$, $A = 3D^2/4$, or $3C^2/36$.

Though not explicitly mentioned, it seems fair to assume[21] that the inherent value of Π in the **Vedas** is 3. The same value of π is used in the **Mahābhārata** (Bhīṣma Parva) and the **Vāyu Purāṇa**[22]. The **Baudhāyana Śulba Sūtra** also imply the same approximation. Brahmagupta takes the same value of π for gross calculaton of area and circumference of a circle[23].

Ex.5 "O learned mathematician, tell the gross circumference and the area of the circle whose diameter is 10, for day-to-day use."

GROSS CIRCUMFERENCE OR AREA OF A CONCH FIGURE

R.10-11 "In a conch figure, multiply (the difference of) the diameter less a half of the face by 3. (The product) is the circumference. Multiply the product of (the face and) the circumference by 1/12. (Subtract) the product from the square of a' half of the diameter'. The difference multiplied by 3 is the (gross) area. Thrice the sum of squares of halves of 'the difference of the diameter as lessened by a half of the face' and 'a half of half of the face' is the gross area of a conch figure.[24]"

Let A be the gross area, F, the face, C, the circumference and D, the diameter of a conch figure. According to the rule,

$$C = 3\left(D - \frac{F}{2}\right), \text{ and } A = 3\left[\left(\frac{D}{2}\right)^2 - \left(\frac{FC}{12}\right)\right]$$

$$= 3\left[\left\{\frac{1}{2}\left(D - \frac{F}{2}\right)\right\}^2 + \left(\frac{1}{2}\cdot\frac{F}{2}\right)^2\right]$$

Ex. 6 "O learned, tell the circumference and the area of the conch circle whose face is 8 and, central diameter, 24."

GROSS ARC OR AREA OF THE SEGMENT OF A CIRCLE

R. 12 "Add the smaller between the chord and twice the height to twice the greater. A half of the sum is (approximate) arc. The result multiplied by a fourth of the smaller is the (gross) area of the segment of a circle."

Let c be the length of the chord, h, the height and a, (the approximate) length of the arc and A, (the gross) area of the segment of a circle. According to the rule.

$$a = \left(\frac{4h + c}{2}\right), \text{ or } (c + 2h)$$

and $A=\frac{C}{4}\left(\frac{4h+c}{2}\right)$, or $\frac{h}{2}(c+h)$, according as $2h > c$, or $c > 2h$.

Mahāvīra [25] and Śrīdhara [26] have also given rules for the area of a segment of a circle.

Ex. 7 "Tell the arc and the area of the segment (of a circle) where the chord is 10 and the height, 4 and (also where) the chord is 12 and the height,8."

AREA IN SOME OTHER CASES

R.13 "[Figures resembling] an elephant's tooth (and a part of) an annular figure happen to be a triangle (and) a quadrilateral (respectively). (Those resembling) a crescent moon, a barley and a **vajra** happen to be two triangles (and those resembling) a **ḍhakka** and a **mṛdaṅga**, two quadrilaterals, separately[27]."

According to the rule, for gross calculations, certain figures should be supposed to be a triangle, a quadrilateral or a pair of them as the case may be.

Ex.8 "The measure of the width of a figure resembling an elephant's tooth is 5 and the other two sides are 16 and 12. A part of an annular figure has sides 15 and 9 and the ordinate, 2. What are their areas."

Ex.9 "In (a figure resembling) a crescent moon, the perpendicular is 3 and the sides are 20 and 16. In a barley shaped figure, the perpendicular is 12 and the sides are 30 (each). What are their areas."

According to Nārayaṇa in the latter case, area of such a figure can also be obtained by considering it as the sum of two segments of circles. Chord in such a case can be obtained from the length and 'the height of the arc' from R.12 by the reverse method.

Ex. 10 "Sides of (figures resembling) a **vajra**, a **ḍhakka** and a **muraja** are 16 (each). Measures of the faces are 4 (each) and lengths in the middle are 0, 1 and 6, respectively. O friend if you know, tell the gross areas (of these figures) produced from circles."

According to Nārāyaṇa in other cases also, the gross area should be obtained by dividing the figure into other similar figures.

GROSS AREA OF A WHEEL LIKE FIGURE

R.14 "Add the square of the width to the product of the width and the diameter of the inner circle. Thrice the sum happens to be the area of a wheel like fugure[28].

Here a piece of the wheel of a chariot is (called) **Nemi.**"

Let D and d be the diameters of the outer and the inner circles of a wheel, respectively and w be its width. The rule states that (gross) area of the wheel = 3 $(dw + w^2)$, which follows by taking $\pi = 3$, and $D = d + 2w$.

Ex.11 "O friend, tell me (the gross) area (of a figure resembling) the wheel of a chariot for day-to-day use (whose) width is 2 and the diameter of (whose) inner circle is 8."

GROSS AREA OF A REGULAR POLYGON

R. 15 "Subtract the number of sides from the square of the number of sides. Multiply (the difference) by the square of the side. (The product) divided by 12 is the (gross) area of a triangle etc.[29]"

Let n be the number of sides, each of length s, of a regular polygon. According to the rule, gross area, A, of the polygone = $(n^2 - n)\, s^2/12$.

Ex. 12 "O learned, if you are an expert in mathematics, tell the areas of (regular) figures (whose) sides are 6 (each) and the number of sides, 3 etc. upto 6."

GROSS AREA OF THE CURVILINEAR FIGURE

R.16 "Multiply the square of a half of the sum of diameters by the number of circles less one. (The product) divided by 9 times the number of circles is the (gross) area (of the curvilinear figure formed by circular arcs) inside (the polygons)[30].

The rule gives the area of the curvilinear figure formed by circular arcs when the lines joining the centres of the circles form a regular polygon. According to the rule, the desired area = $(nd/2)^2$. (n-1)/9n, d being the diameter of the circles and n being the number of circles.

Ex.13 "Tell (the curvilinear) areas inside the regular figures of sides starting from 3 upto 6, within the connected circles, the diameter of (each of the circles) being 12."

GROSS DIAMETER OF A CIRCLES.

R.17 "Area of the circle is added to (its) one-third. Square-root (of the sum) is the diameter."

According to the rule, $d = \sqrt{A + \frac{A}{3}}$, d being the diameter and A, the area of the circle. The rule is obivious as gross value of $\pi = 3$.

Ex.14 "O friend, if you know tell quickly the diameter of the circle whose area is 75."

Gross Diameter or Face of a Conch Figure

R.18-19a "In a conch figure, divide the area by 3. (Set the quotient in two parts). Double the square-root of the greater and (also), the (the square-root of the lesser, which remains. Their sum is the diameter (and) twice the lesser is the face."

Let A be the area of a conch figure whose face is a and diameter, d. Divide $A/3$ in 2 parts say m and n and for definiteness suppose that $m > n$. According to the rule, $d = 3(\sqrt{m} + \sqrt{n})$ and $a = 2n$; Here, it may be noted that A/3 can be set as the sum of 2 numbers in several ways. So, from a given area several conch figures can be formed. The rule is converse of rule10-11.

Ex.15 "O friend, if you are an expert in mathematics, tell quickly the diameter and the face of the conch figure whose area is 60."

Side of a Regular Polygon

R.19b "Multiply the area by 12. Divide (the product) by the difference of the difference of 'the square of the number of sides' less 'the number of sides'. Square-root of the product is (the length of) the side (of a regular polygon)."

With the notation of R.15, according to the rule, $s = \sqrt{12A/(n^2 - n)}$

Ex.16 "In a triangle, a square, a pentagon and a hexagon, the areas are 8,25,60 and 40, respectively. Tell (the lengths of) their sides."

EXPLANATION

R.20 "The unknowns should be obtained from the formula from which (the respective) results have been obtained, by the reverse process. The same has not been told by me for fear of being exhaustive."

Bases of Partial Figures

R.21 "Multiply the difference between the base and the face by the partial flank side. (The product) divided by the sum of partial flank sides happens to be the common difference. The common differences added to the respective faces are the respective bases (of the partial figures)."

Rational : Let B, F and S denote the base, the face and a flank side of any quadrilateral and b, f and s those of a partial part of it, respectively. According to the rule, the common difference (= d, say) $(B - F)s/S$ and $b = f + d$.

Let CEGH be a trapezium (Fig.1) and cegh, a partial part of it such that ch II eg II CH. Let CP and cṗ be parallel to HG. Suppose that base EG=B, face CH=F, flank side CE = S and eg=b, ch=f and ce=s. As CP II cṗ II HG, so, EP=B-F and $e\dot{p}$ = b-f. From similar triangles CEP and $ce\dot{p}$. We have, $\frac{ce}{CE} = \frac{ep}{EP}$ or $s/S = (b - f)/(B - F)$

From this, we get $b = f + \frac{(B-F)s}{S}$ and hence the rule.

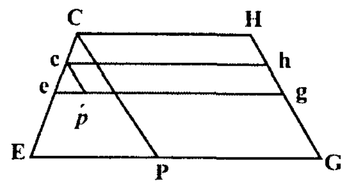


Fig. 1

Ex. 17 "Measures of the face, the base and (each of) the flank sides of a figure having 6 parts are 1, 7 and 14 (respectively). Tell the bases of the middle parts. What will be the separate areas if the measures of (each) part of the flank side be 48"

Ex. 18 " Measures of the face, the base and (each of) the flank sides, (of a figure) are 2, 9 and 28 (respectively). Flank sides of the first etc. (parts) are 8, 16 and 4 (in order). Tell me the bases of the middle parts."

Ex.19 "Measures of the base, the face and the flank sides (of a figure) are 8, 2, 54 and 42 (respectively). Starting from the base (upwards) the two flank sides of the three parts are 9, 7; 18, 14 and 27, 21 (in order). O learned, if you know mathematics tell quickly the bases of the middle parts."

BASE AND FLANK SIDES OF A PARTIAL FIGURE

R. 22-23 "Multiply the difference between the base and the face by the partial area. Divide (the product) by the sum of flank sides. Add (the quotient) to the square of a half of the face of the partial figure. Twice (the sum) happens to be the base of that partial figure.

Multiply the difference between the base and the face of the partial figure by a flank side. Divide (the product) by the difference between the base and the face. (The quotient) happens to be (the corresponding) gross (and also) the exact side of that partial figure.[31]"

In Fig.1, let us further suppose that the flank sides $HG = R$, $hg = r$ and the area of cegh $= a$.

According to the rule, $b = 2.\left[\frac{(B-F)a}{R+S} + \left(\frac{f}{2}\right)^2\right]^{1/2}$, $r = \left(\frac{b-f}{B-F}\right)R$

and a similar expression for s.

Rational : From similar triangles $ce\dot{p}$ and CEP, we have, $e\dot{p}/EP = ce/CE = c\dot{p}/CP$, or $(b-f)/(B-F) = s/S = r/R$, from which we get the second part of the rule

or $(b-f)/(B-F) = (r+s)/(R+S)$

or $b - f = \frac{(B-F)(r+s)}{R+S}$

or $(b-f)(b+f) = \frac{4(B-F)\left(\frac{r+s}{2}\right)\left(\frac{b+f}{2}\right)}{R+S}$

or $\left(\frac{b}{2}\right)^2 - \left(\frac{f}{2}\right)^2 = \frac{(B-F)a}{R+S}$

or $\left(\frac{b}{2}\right)^2 = \frac{(B-F)a}{R+S} + \left(\frac{f}{2}\right)^2$ from which the first part of the rule can be esily derived.

Ex. 20 "Measures of the face, the base and (each of) the flank sides of a figure 4, 10 and 24 (respectively). Areas of the three parts (of the figure) are 40, 56 and 72 (in order. Tell the two sides (which lie) in the middle (i.e., the base and the face of the middle part) and both the flank sides of (each of) the three parts."

Ex. 21 "Measures of the face and the base (of a figure) are 2 and 8 (respectively). (Its) flank sides are 54 and 42. Measures of the separate areas of (its) 3 parts are 84, 96 and 60. Tell me quickly the measures of the sides (which lie) in the middle (i.e., the base and the face of the middle part) and (both) the flank sides of (each of) the three parts."

Ex. 22 "The earlier gross rules have been stated for novice calculators. Due to occasional disagreement between (gross and exact) results I have not much respect (for them)."

EXAMPLE TO SHOW DISAGREEMENT BETWEEN GROSS AND EXACT RESULTS.

Ex.23 "The bases of two figures are 360 and 38, the (respective) altitudes are 19 and 180 and (each) of the two flank sides of both (the figures) is 181. Look here, whether the gross area is to be accepted or discarded."

Here exact area of both the triangles are equal whereas their gross areas are 32580 and 3439, respectively.

EXACT METHODS

R.24-27 "For a square, the length is the upright and the breadth is the side. When the lenght is the side the breadth happens to be the upright. Mathematicians have named Length and Breadth for practical purposes. Here, the difference is only in name and, not in nature.

In a square or a rectangle, the square-root of the sum of squares of the upright and the base is the diagonal. The square-root of the difference of the squares of the diagonal and the upright happens to be the side[32]. Half of that (i.e., of a square or of a rectangle) made by a diagonal happens to be the triangle of the quadrangle (i.e., a right angled triangle)."

Here the rule given contains the so-called 'Pythagoras Theorem,' although it was known in Babylonia, China, as well as in India, much earlier than the time of Pythagoras[33] (6th century B.C.). Dr. Shukla has pointed out that the theorem is stated in the **Garga-Saṁhitā** of Garga as quoted in a commentary, written in the 16th century, on the *Āryabhaṭīya* of Āryabhaṭa I.[34] Garga is said to have been born in the beginning of Kaliyuga which started[35] in 3102 B.C. Recently it has been shown by A.Seidenberg that the **Taittirīya Saṁhitā** (before 2000 B.C.?) knows not only the algebraic or computational aspect of the theorem, but also the geometric or constructive aspect which was not known to the Babylonians of about 1700 B.C.[36] Lastly, we also observe that a right-angled triangle is treated as part of a rectangle as has been the tradition in Hindu mathematics.[37]

Ex. 24 "Where the measures of the upright and the side is 3 and 4 (respectively), tell the diagonal. O mathematician, (tell) the upright from the diagonal and the side and the side from the diagonal and the upright."

Operations for Radicals

R.28-30a "The square-root of a number, if not an exact number, is called the surd of the number. In case of multiplication or division (of a surd) by a number. In case of multiplication or division (of a surd) by a number, multiply or divide (the number under radical sign) by the square (of the number). Divide the greater number (which is under radical sign) by the smaller one (Extract) the square-root (of the quotient). Add one (to the quotient and also) subtract one (from it). Multiply (the results) by themselves. Multiply (the squares) by the smaller number. The products under the radical signs are the sum and the difference (of the numbers) under radical signs (in order). If the quotient has no (exact) square-root, the numbers under radical sign exist (separately) as the sum or the difference (as the case may be).[38]"

First part of the rule defines a surd. According to second part of the rule,

$$a \times \sqrt{b} = \sqrt{a^2 b}, \text{ and } \sqrt{b}/a = \sqrt{b/a^2}.$$

Next, suppose we have to find, $\sqrt{a} \pm \sqrt{b}$.

If a/b is a square number (c^2, say) so that $c = \sqrt{a}/\sqrt{b}$, we have, according to third part of the rule, $\sqrt{a} + \sqrt{b} = \sqrt{b(c+1)^2}$ and $\sqrt{a} - \sqrt{b} = \sqrt{b(c-1)^2}$

Rational : $\sqrt{b(c+1)^2} = \sqrt{b\left(\frac{a}{b} + 1 + 2\frac{\sqrt{a}}{\sqrt{b}}\right)}$

$= \sqrt{(\sqrt{a} + \sqrt{b})^2}$

$= \sqrt{a} + \sqrt{b}$, and similarly for $\sqrt{a} - \sqrt{b}$ and hence the rule.

According to last part of the rule, if a/b is not a square number, the sum and the difference will remain in the form of radicals, as they are.

Several other rules for the addition and the subtraction of surds have been given by Nārāyaṇa in his book on algebra, the **Bījagaṇitāvataṁsa.**[39]

Approximate Value of a Surd

R.30b-31a "Multiply (the numerator of) a number under radical sign by (its) denominator. Multiply (the product) by a great square number such as 100 etc. (Extract) the square-root (of the final product). Divide (the square-root) by the product of 'the root of the multiplier' and 'the denominator, The quotient is an approximate value of the surd."[40]

According to the rule, $\frac{\sqrt{abn^2}}{bn}$ is an approximate value of $\sqrt{\frac{a}{b}}$, where n^2 is a great square number. Naturally, the greater the squaree number, the better will be the approxmiation. Clearly, the denominator of an integer[41] is 1.

Ex.25 "O learned, a side of a square is 6. Tell me quickly, (its) diagonal. (Also) an upright and a side of a rectangle is $3\frac{1}{3}$ and $5\frac{1}{3}$. Tell (its diagonal)."

EXACT METHODS (CONTINUED)

R.31b-32a "One should obtain (the desired) by one's own intelligence applying the rule of concurrence between the sum and the difference of any two out of the side, the upright and the diagonal."[42]

The Bamboo Problem

Ex. 26 "The measure of a bamboo is 36 cubits. It is broken by wind. The distance between the tip of the bamboo and its base is 24 cubits. Tell (the lenghts of) the parts of the bamboo.[43]"

The Arrow Problem

Ex.27 "In a battle, the height of the elephant is 14 cubits. An archer is at a distance of 98 cubits. The person flung an arrow for cutting the hand (of the enemy). The warrior liberated himself by flinging his arrow quickly. His arrow destroyed the arrow (of the enemy). At how many cubits in the way from the elephant, (the arrows) met. Tell that."

The Lotus Problem

Ex.28 "In a pond full with cranes moving with lapping waves of water, a budding lotus, while at rest like a living being, is at a height of 1 cubit (above

water). The lotus is (just) drowned by the slow wave of wind after moving 7 cubits. Tell me the depth of water and the height of the lotus stalk.[44]"

Similar examples occur in a Chinese work called **Chiu-chang Suan-shu.** However, the chinese solution to the problem is quite different from that of Bhāskara I (629A.D.).[45]

THE CUCKOO PROBLEM

The problem is as follows. Two cuckoos are sitting at the ends of a diagonal of a pond. One cuckoo moves along a side of the pond and the other along an oblique line intersecting the above line. They start simultaneously with the same speed and meet at the point of interesection of the above lines.

R.32b-33a. "Multiply the square of the diagonal by the side to be gone. (Also, add) the square of the diagonal to the difference between the squares of the sides adjacent to the diagonal. (The above product) divided by the sum is equal to the distance (between the place) of union of the cuckoos (and the point of start)."

Let the horizontal section of the pond be in the form of a quadrilateral whose diagonal is d, the side to be gone, b and the other side, a. According to the rule, the

distance moved = $\dfrac{d^2 b}{d^2 + (b^2 - a^2)}$.

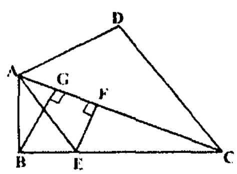


Fig. 2

Rational : Let ABCD (Fig.2) be a horizontal section of the pond. Suppose that $AC = d$, $BC = b$ and $AB = a$. Let one cuckoo be at A and the other at C and suppose that one of them move along the oblique line AE and the other, along the side CB to meet at E. Let EF and BG be perpendiculars to AC. We have

$$GC = \frac{(GC + AG) + (GC - AG)}{2}$$

$$= \frac{AC}{2} + \frac{(GC - AG)(GC + AG)}{2(GC + AB)}$$

$$= \frac{AC}{2} + \frac{(GC^2 + BG^2) - (AG^2 + BG^2)}{2AC}$$

$$= \frac{AC^2 + BC^2 - AB^2}{2AC} \qquad \ldots (1)$$

But $GC = (FC.BC)/EC$ from similar triangles EFC and BGC;

$$= (AC/2) \,.\, BC/EC \qquad \ldots (2)$$

From (1) and (2), the result can be easily obtained.

Ex.29-31 "The lengths of the northern and the southern flanks of a pond are 16 cubits (each) and the breadths of the eastern and the western flanks are 12

cubits (each). The male cuckoo, out of the joyously moving pair, separated in the night, is at the upper east corner. The female cuckoo, at the uppersouth corner. The female cuckoo, at the upper south corner, moved on the south side. Seeing her, the male cuckoo moved quickly in an oblique direction to meet her for coitus. (Both of them travelled) the same distance. If you know mathematics, tell that."

AREA OF A TRAINGLE AND, A (CYCLIC) QUADRANGLE

R.33b-34a "Put half the sum of sides at 4 places. Subtract the sides (from these). The square-root of the product (of the remainder) is the exact area of a triangle and, occasionally, of a quadrangle".[46]

Let a, b, c and d be the sides of a quadrangle and suppose that $s = (a+b+c+d)/2$. The rule states that, occasionally, the exact area of a quadrangle $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

In the case of a triangle, one of the sides becomes a cipher and the area is always exact. In the case of a quadrangle since the area is only occasionally exact, here Nārāyaṇa criticizes Ācārya Śrīdhara as the latter has not stated the limitation of the rule.

Ex. 32 "Tell me quickly the exact areas of the square, the rectangle and also of triangles stated in the rules on rough (areas)."

LIMITATION OF THE RULE

Ex.33 "What is the area (of the qudrangle) in which the base is 21, the measures of flank sides are 10 and 17, the face is 12 and the altitude, 8."

Here, Nārāyaṇa takes the quadrangle as a trapezium in which, the face is elongated towards one side and the base, towards the other. He divides the trapezium in 3 parts, 2 right angled triangles with sides ,10, 6 and 8 and 17, 15 and 8 a rectangle with sides 6 and 8 and thus finds the exact area of the quadrangle as 132. However, by the above rule, its area will be $\sqrt{42120}$, which is not true.

AREA OF A TRAPEZIUM

R.34b "Add the base to the face. Half (of the sum) multiplied by the altitude is the area of a trapezium."[47]

Let a, b, h and A be the face, the base, the altitude and the area of a trapezium, in order. According to the rule, $A = (a+b)h/2$. A triangle is considered to be a trapezium whose face is a cipher.

Ex. 34. "O learned, if you have the least of your pride, tell the exact area of the (trapezium-like) figure whose face is 9, base is 21, flank sides are 10 (each) and the measure of altitude is 8."

Ex.35 "Tell me the area of the triangle whose altitude is 8, the measures of (whose) flank sides are 10 and 17 and the base is 21."

EXACT AREA OF A CIRCLE ETC.

R.35 "In circles etc., in regular polygons having 3,4 etc. as the number of sides and also in circular figures encircled by regular polygons having 3 etc. sides, the gross (result) multiplied by 527 (and then) divided by 500 is the exact (result).

Rules for obtaining gross areas of the above figures have been given in rules 9 to 16, in order. Obviously, here Nārāyaṇa considers the value of Π as 3x527/500 i.e., 3.162. This seems to be from the Jain value[48] of π which is $\sqrt{10} = 3.162...$ R.C. Gupta has dealt with Jain Value of π, its popularity[49] in India along with its transmission[50] to foreign countries and specially in countries [51] such as China and Japan and its popularity [51] in these countries. He has also dealt with other values of π prevalent[52] in India and its transmission[53] to other. countries.

Ex.36-37. "O learned, if you know mathematics,tell me the exact area (etc.) of the circle, the conch figure, the arc (of a circle), the diamond (shaped figure), the annualr figure, (the figure resembling an elephant's) tooth, (the figures resembling) a crescent moon, a barley corn, a dhakka, a mṛdaṅga, a wheel, the regular polygons having 5, 6 etc. sides and the circular figures encircled by regular polygons having 3 etc. sides stated under the gross method."

PROPERTIES OF TRIANGLES AND QUADRILATERALS

R. 36-39. "In a traingle, divide the product of the sum and the difference of two flank sides by its base. Set down the quotients in two places. (At one place) half of the sum of (the quotient) and the base (and at the other place, half of) the difference (of the quotient) and the base are its segments[54]. The names of the smaller and the greater segments are **Sandhi** (the link) and **pītha** (the complement of the segement), respectively and they lie towards the lesser and the greater flank sides from the altitude, (in order). The square-root of the difference of the squares of a flank side and its own segment is the altitude[55]. The sqaure-root of the difference of the squares of the flank side and the altitude is the segment (corresponding to that side). The square-root of the sum of squares of the altitude and a segment is the flank side (corresponding to that segment). Half of the altitude multiplied by the base is the exact area of the traingle.[56]"

Here Segments mean segments of the base by the altitude. In the griantle ABC Fig.3, let $CD = (p)$, be the altitude,

$AD = n$, $DB = m$ and $m > n$. According to the rule,

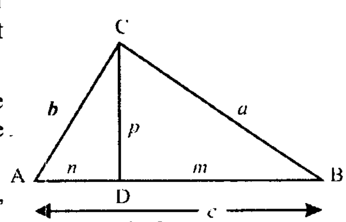


Fig. 3

(1) $m = \frac{1}{2}\left[c + \frac{(a^2 - b^2)}{c}\right]$ and $n = \frac{1}{2}.\left[c - \frac{(a^2 - b)^2}{c}\right]$

(2) $p = \sqrt{a^2 - m^2} = \sqrt{b^2 - n^2}$.

(3) **Sandhi** $(= n) = \sqrt{b^2 - p^2}$ and **pīṭha** $(= m)$ $\sqrt{a^2 - p^2}$

(4) $a = \sqrt{m^2 + p^2}$ and **pīṭha** $b = \sqrt{n^2 + p^2}$ and

(5) area of the triangle $ABC = p.c/2$

For **sandhi** and **pīṭha** in a quadrangle R51 and comments after R.54b-55a may be seen. The above rules can be easily obtained by using R.26-27.

Ex. 38 "If you know, tell quickly the exact area, the measures of the altitude and the segments (of the base made by the altitude) of the triangle the measures of whose flank sides are 13 and 15 and the base, 14."

Ex. 39 "O friend, tell quickly the exact altitude in the triangle, the measures of whose sides are 20 and 13 and, the base 11. (Tell) the flank sides from the altitude and the segments and, the segments of the base from the altitude and the flank sides."

Here, using relation [(1), second part] the segment obtained is - 5 (a negative quantity). According to Nārāyaṇa, in such cases, the perpendicular from the vertex to the base of a triangle does not lie within the area of the triangle as specified by its sides.

Characteristic of a Figure to Exist

R.40 "In a quadrangle or a triangle in which the sum of other sides is itself smaller than or equal to the greater side, the figure does not exist.[57]

"Ex.40 "Mesures of the flank sides of a quadrilateral are 5 and 9, its base is 18 cubits and the face, 4. The flank sides of a triangle are 6 and 15 and its base is 27. If you know geometry and are an expert in the methods of mathematics, show here quickly whether the areas of these figures, badly joined by straight lines, exist or not."

Here, according to Nārāyaṇa, the rule given by Śrīdhara for obtaining the perpendicular and the segments is not true Nārāyaṇa quotes the latter's following rule in this respect.

"Only in those (quadrilateral) figures, in which the square of the base minus the face exceeds the product of the difference and sum of the flank sides do' the perpendiculars dropped from the vetices to the base' (lamba) and 'the segments into which the base is divided by them' (abadhā) exist.[58]"

Consider Ex.40 given above. According to Nārāyaṇa if we apply the rule as given by Śrīdhara, here in this case the perpendicular is obtained and due to this, the area is also obtained. But since, the figure does not exist, the area does not exist and hence the rule is wrong.

Also, according to Nārāyaṇa, the rule given by Bhāskara II for obtaining the perpendicular (from the vertex to the base) of a triangle is badly framed. Nārāyaṇa quotes the latter's following rule in this respect.

"In a triangle, the sum of two sides, being multiplied by their difference is divided by the base; the quotient is subtracted from, and added to the base which is twice set down; and being halved, the results are segments corresponding to those sides. The square-root of the difference of the squares of the side and its own segment of the base becomes the perpendicular Half the base multiplied by the perpendicular is, in a triangle, the exact area.[59]"

Consider Ex. 40 given above. According to Nārāyaṇa, if we apply the rule as given by Bhāskara II we get the segments as 10 and 17 and so the perpendicular is obtained as 8 and therefore the area of the triangle is got as 108.

Now, if we apply Nārāyaṇa's rule, according to him, the perpendicular is obtained as $\sqrt{-64}$ and since -64 is a non-square, it has no square-root and so the perpendicular can not be got. Due to this the area also cannot be obtained and therefore, the formation of a trianble is not possible and hence the cirticism of Bhāskara II's above rule by Nārāyaṇa.

Here, we also note that if we apply R. 33b-34a to Ex.40, the area of the quadrilateral becomes zero and that of the triangle cannot be got as it becomes the square-root of a negative number.

Relation between sides, Diagonals and Area of a Quadrilateral

R.41-45 "A quadrangle whose both diagonals are indeterminate has indeterminate area nad even with the same sides but with different diagonals, it has different area.[60] With the same sides if one diagonal is shortened, the other lengthens and according to one's supposition, diagonals increase or decrease.

(For finding) the desired (second) diagonal with the (or given) first diagonal, form two triangles with the (assumed or given) diagonal as their hases and the flank sides (of the quadrangle) on the same side of the base as their sides, separatey. Now, multiply the difference between two segments (of the base) lying on the same side of the point of interesection (of the diagonal and) the perpendicular from the vertex of the triangle) by itself. Add the square of the sum of perpendiculars to the product. Square-root of the sum is the other diagonal.[61]

Half of a diagonal multiplied by the sum of perpendiculars (from the vertices to the diagonal) is the area (of the quadrangle). In a square and rectangle, the product of a side and an upright is the area (and they have) equal diagonals.[62]"

In Fig4., let AC (=e) and BD be the diagonals of a quadrilateral $ABCD$ whose area is A'. Suppose that BE (= p) and DF (= q) are perpendiculars from B and D, respectively to AC. Let EF = r. Accordingly $BD = \sqrt{r^2 + (p+q)^2}$ and $A' = (p+q)\, e/2$.

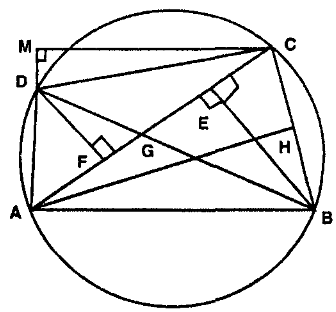


Fig. 4

Ex. 41 "Tell the diagonal and the area of the square whose side is 65 and also of the rectangle whose upright is 75 and the side, 100."

Ex. 42-43 "The measures of flank sides (of a quadrangle) are 40 and 68, its face is 51 and the base, 75. Supposing one of its diagonal, tell the other. O friend, by interchanging its smaller flank side and the face, tell (again) the other diagonal."

RESTRICTION ON THE ARBITRARY ASSUMPTION OF DIAGONAL

R. 46 "Supposing the sum of smaller pair of sides about a diagonal as the base (and the other sides as) the sides of a triangle, obtain the perpendicular (from the vertex of the triangle to its base). The diagonal cannot be greater than the base nor (the other diagonal) can be smaller than the perpendicular[63]."

We have seen that if the sides and a diagonal of a quadrangle be known, the other diagonal can be obtained by R. 43-45. However, if all the sides of a quadrangle be known but none of its diagonals, the quadrangle become determinate only if one amongst its diagonals is assumed arbitrarily. After that, its other diagonal and the area can be determined. However, the diagonal cannot be assumed to be arbitrarily large or small. The rule restricts the arbitrariness of such an assumption.

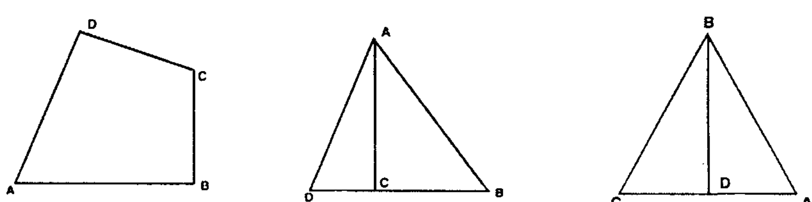


Fig.5

Let ABCD (Fig.5) be a quadrangle. Let two triangles ADB and BCA be formed such that '$DB = DC + CB$ and AC is perpendicular to DB in the former triangle and '$CA = CD + DA$ and BD is perpendicular to CA' in the latter. According to the rule,

(i) perpendicular $AC \leq$ diagonal $AC \leq$ sides $(CD + DA)$

(ii) perpendicular $BD \leq$ diagonal $BD \leq$ sides $(BC + CD)$.

Ratonal : Let the quadrangle ABCD be deformed in such a manner that the sides remain the same but the diagonals change. Naturally as one diagonal increases the other decreases. Suppose DB goes on increasing. Obviously AC will go on decreasing. In the extreme case, when DB takes its maximum value the quadrangle ABCD takes the shape of the traingle ABD in which the diangonal DB takes the shape of the base DB. Now the diagonal AC will become any line joining A to a point C' in DB. But the smallest line joining A to a point in DB is the perpendicular AC. This establishes the second part of (ii) and first part of (i). Similarly the other parts can be established.

Diagonals of a (Cyclic) Quadrilateral

R.47-52. "Divide the sums of the products of the sides about both the diagonals by each other. Multiply the quotients by the sum of the products of opposite sides. Square-roots (or the) products are the diagonals in quadrilateral.[64]

In all (cyclic) quadrilaterals, the (new) diagonal obtained by the interchange of its face and flank side is the third diagonal. Thus, there are three diagonals in (cyclic) quadrilateral. In equilateral and equitrilateral quadrilaterals, the three diagonals are equal. In equidichastic (a=c, b=d) and equibilateral, only two diagonals are equal. In a scalane quadrilateral, (the diagonals are) unequal. In triangles in a quadrilateral diagonals are the bases and base and sides are the sides.

In all quadrilaterals, perpendiculars and segments should be obtained as in a triangle. In a triangle, product of sides (about a perpendicular) divided by the perpendicular is the diameter (of the circumcircle).[65] Product of the three diagonals divided by twice the (circum) diameter is the area (of the cyclic quadrilateral). In a triangle or (a cyclic) quadrilateral half of the (circum diameter is the (circum) radius."

According to the rule. (see Fig.4),

$$AC\ (= e) = \left[\frac{(ac + bd)\ (ab + cd)}{(ac + cd)}\right]^{1/2} \quad \ldots.(1)$$

$$BD\ (= f) = \left[\frac{(ac + bd)\ (ad + bc)}{(ab + cd)}\right]^{1/2} \quad \ldots.(2)$$

If face c is interchanged with the flank side d, in place of AC, we get another digonal say

$$g = \left[\frac{(ab + cd)\ (ad + bc)}{(ac + bd)}\right]^{1/2} \quad \ldots.(3)$$

If c is interchanged with the flank side b, the same is obtained in place of BD. Nārāyaṇa calls this as the third diagonal of the (cyclic) quadrilateral ABCD.

In all quadrilaterals, perpendiculars and segments should be obtained as in a triangle by R. 36-39. Circum-diameter of the quadrilateral is to be obtained from a triangle whose base is a diagonal of the quadrilateral and whose flank sides are a flank side and the base (or face) of the quadrilateral.

Thus, circum-diameter of the quadrilateral ABCD (= D), = $ab/p = cd/q$ and circumradius D/2. Also, area of the quadrilateral = $efg/2D$.

From (1), (2) and (3), we observe that in equilateral ($a=b=c=d$) and equitrilateral ($b=c=d$) quadrilaterals, all the three diagonals are equal and these are equal to $\sqrt{2}\,.a$ and $\sqrt{b(a+b)}$ respectively, similarly, in equidichastic ($a=c$, $b=d$), and equibilateral ($b = d$) quadrilaterals, 2 diagonals are equal and these are equal to $(a^2 + b^2)^{1/2}$ and the third diagonal (= g) is equal to $\frac{2ab}{(a^2+b^2)^{1/2}}$ and $\frac{ab+bc}{(ac+b^2)^{1/2}}$ in order.

Rational : In Fig. 4 let AH be the perpendicular from A to BC, and CM, that from C to AD (extended, if necessary). We have,

$$e^2 = AH^2 + HC^2 = AH^2 + HB^2 + BC^2 - 2HB.BC$$
$$= a^2 + b^2 - 2ab \operatorname{Cos} B \ldots (1)$$

Similarly,

$$e^2 = c^2 + d^2 - 2ab \operatorname{Cos} B \ldots (2)$$

Multiplying (1) by cd and (2) by ab and adding, we get

$$(ab + cd)\, e^2 = (a^2 + b^2)\, cd + (c^2 + d^2)\, ab.$$
$$= (ac + bd)(ad + bc).$$

$$\text{So,} \quad e = \left[\frac{(ac+bd)(ad+bc)}{(ab+cd)}\right]^{1/2}$$

Similarly f can be derived. Bhāskara II has quoted the rule given by Brahmagupta for finding the diagonals of a quadrilateral.[66] The formula is stated to be rediscovered in Europe by W.Snell (about 1619 A.D.)[67].

Ex. 44 "O learned, determine quickly, the two diagonals and by interchanging the face and a side, determine the third diagonal of the square (and) the rectangle stated earlier.[68]"

Ex. 45 "Tell the two segments, the diagonals, the perpendiculars, the (circum) diameter and the circum) radius of the figure whose flank sides are 25 (each) base is 17 (and), face, 3."

Ex. 46 "Tell the two segment, the diagonals, the perpendiculars, the (circum radius (of the figure whose) flank sides are 25 (each), face is 25 (and) base,39."

Ex. 47 "O friend, tell the two segments, perpendiculars, (the diagonas), the (circum) diameter, the area and the (circum) radius (of the figure) whose flank sides are 390 (and) 520, base is 600 (and) face, 250."

Ex. 48 "If you know, tell quickly the measure of the circumcircle of the triangle, measures of whose flank sides are 13 (and) 15, base is 14 and perpendicular (from the vertex to the base)12."

SEGMENTS AND PERPENDICULAR OF A TRAPEZIUM

R.53-54a "In a trapezium, (supposing its) base less face as the base (and) its two flank sides as the two flank sides of a triangle, its two segments and the perpendiculars should be obtained by the method stated earlier[69]. Add the square of the sum of the segment and the face to the square of the perpendicular. Square-root of the sum is the diagonal."

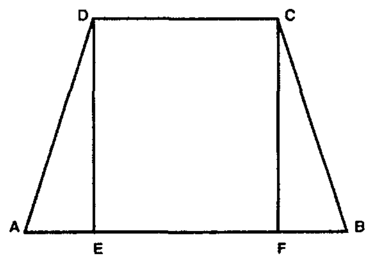


Fig. 6

Consider the trapezium ABCD (FIG.6), DE and CF are perpendiculrs to AB. According to the rule,

$$AC=\sqrt{(AE+DC)^2+CF^2}, \text{ and } BD=\sqrt{(DC+FB)^2+CF^2}$$

Ex.49 "O mathematician, tell quickly the two segments, diagonals and perpendicular of the equibilateral, equitrilateral (and scalene) trapeziums stated earlier[70].

SŪCĪ KṢETRA : SEGMENTS OF DIAGONAL AND PERPENDICULAR

R. 54b-55a. "Multiply one's own **sandhi** by the other perpendicular and by one's own diagonal (severally). The products divided by the other **pīṭha** (severally) are the lower portions of the perpendicular and, of the diagonal below the point of intersection of one's own perpendicular and, (own) diagonal (respectively)[71].

In the quadrilateral $ABCD$ (Fig.7) let AC and BD be its diagonals, DE and CF be perpendiculars from D and C, respectively, to the base AB and G and H be their points of intersection. Here, AE is the **sandhi,** EB is the **pīṭha** and AF is the other **pīṭha** for the perpendicular DE; and FB is the **sandhi,** AF is the **pīṭha** and EB is the other **pīṭtha** for the perpendicular CF. According to the rule,

(1) $GE=(AE.CF)/AF$

(2) $AG=(AE.AC)/AF$,

(3) $FH = (DE.FB)/EB$, and $HB = (FB.DB)/EB$

Rational : The first two results follow from the similarity of the triangles AGE and ACF and the last two, from the similarity of the triangles HFB and DEB.

Ex. 50 - 52a "If you know the whole of mensuration, tell (the lower) portions of the perpendiculars (and) the diagonals below the points of intersection of the perpendicular (and the diagonal, the perpdndicular below the point of intersection (of the diagonals), the parts of the base made by that, **sūcī** perpendicular, **sūcī** segments (of the base) and **sūcī** sides, in the scalene quadrilateral stated earlier[72]."

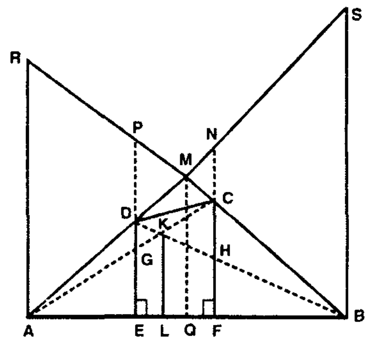


Fig. 7

PERPENDICULAR TO THE BASE AND SEGMENTS OF IT.

R. 55b-56a. "Divide the **pīṭhas** by their own perpendiculars (separately and add the quotients). Divide the base by that sum. (The quotient) is (the lenght of) the perpendicular from the (point of) intersection of the diagonals to the base. That multiplied by them (i.e., by the earlier quotients) are the segments of the base[73]."

In Fig.7, let KL be the perpendicular from the point of intersection of the diagonals to the base. According to the rule,

(1) $KL = AB \left(\frac{EB}{DE} + \frac{AF}{CF} \right)$

(2) $LB = (EB.KL)/DE$

(3) $AL = (AF.KL)/CF$

Rational : From similar triangles, KLB and DEB, We have, $LB/KL = EB/DE$

Similarly, from similar triangles, KAL and CAF, we get $\frac{AL}{KL} = \frac{AF}{CF}$. Adding these two,

$\frac{AB}{KL} = \frac{EB}{DE} + \frac{AF}{CF}$. Thus we get the said results.

SŪCĪ SIDES, PERPENDICULAR, AND SEGMENTS

R. 56b-58a "Divide **sandhis** by their own perpendiculars (severally), Add (the quotients). Divide them (i.e., the quotients) by the sum (severally. The results) multimplied by the base are the **sūcī** segements. Divide perpendiculars by their own **sandhis** (severally. A quotient) multiplied by the **sūcī** segment is the **sūcī** perpendicular. Now, **sūcī** perpendicular multiplied by the side (and then) divide by the perpendicular happens to be the (respective) **sūcī** side[74]."

Let the side AD and BC (Fig. 7) be extended to met at M and let MQ be perpendicular to AB. Then the figure MAB is called **sūcī** kṣetra, MA and MB are the **sūcī** sides, MQ is the **sūcī** perpendicular and AQ and QB are the **sūcī** segments, According to the rule,

(1) $AQ = \frac{AB(AE/DE)}{(AE/DE)+(FB/CF)}$

(2) $BQ = \frac{AB(FB/CF)}{(AE/DE)+(FB/CF)}$

(3) $MQ = \frac{AQ.DE}{AE} = \frac{QB.CF}{FB}$

(4) $AM = \frac{MQ.AD}{DE}$ and (5) $BM = \frac{MQ.BC}{CF}$

Rational : From similar triangles, AED and AQM, $\frac{AQ}{MQ} = \frac{AE}{DE}$...(1) Similarly from triangles MQB and CFB, $\frac{BQ}{MQ} = \frac{FB}{CF}$...(2) Adding (1) and (2), $\frac{AB}{MQ} = \frac{AE}{DE} + \frac{FB}{CF}$...(3)

Dividing (1) and(2) by (3), separately, the first two relations are obtained, in order. The 3rd relation is easily obtained from (1) and (2) respectively. The next relation is obtained by considering the similarity of triangles AMQ and ADE and the last, from that of BMQ and BCF.

EXTENDED SŪCĪ SIDE AND PERPENDICULAR

R.58b-59a "Multiply one's (own side or) perpendicular by the other's pītha. (The products) divide by one's own **sandhi** are the value of the extended side (or) perpendicular (which are) in the form of hypotenuse or ordinate."

ALTERNATIVE RULE

R.59-b60a " Divide **sūcī** side (or) **sūcī** perpendicular by **sūcī** segment. (The quotient) multiplied by the other **pītha** happens to be the lenght of one's own side (or) the other's perpendicular from their point of intersection."

In Fig. 7, let DE, BC and CF, AD be extended to meet at P and N, respectively According to the former rule,

(1a) $AN = (AD.AF)/AE$; (1b) $NF = (DE.AF)/AE$,

(2a) $BP = (BE.BC)/BF$, and (2b) $PE = (EB.CF)/FB$;

and according to the latter rule,

(3a) $AN = (AF.AM)/AQ$; (3b) $NF = (AF.MQ)/AQ$

(4a) $PB = (MB.EB)/QB$ and (4b) $PE = (MQ.EB)/QB$

Rational : Results 1a, 1b; 2a. 2b, 3a. 3b and 4a, 4b follow from the similarity of triangles DAE, NAF; PEB, CFB; MAQ, NAF and PEB, MQB, in order.

Ex. 52b-c "If you have pride in mensuration, tell me very quickly, the extension in the side and the perpendicular upto (their) point of intersection in the scalene quadrilateral stated earlier (when) the side (and) the perpendicualr are extended (to meet at a point)[75]."

Obviously, extension = **sūcī** side (of **sūcī** perpendicular) - side (or perpendicular).

VAṀŚA, RAJJU, SŪCĪ PERPENDICULAR AND SEGMENTS OF BASE.

R.60b-63a "Multiply one's own (side or) perpendicular by the base (the product) divided by one's own **sandhi** is the (corroesponding) **rajju** or **vaṁśa**. Product of (two) vaṁśas, divide by their sum is the perpendicular, from the point of intersection of extended **rajjus** (i.e., lines) from the root of one (vaṅśa) to the tip of the other (which) are in the form of a diagonal and an upright, to the base[76]. A **vaṁśa** multiplied by (the base and then) divided by their own sum (i.e., sum of the **vaṁśas**) is the segment of the base. That (i.e, a **vaṁśa** multiplied by the **sūcī** side, the product) divided by the **rajju** is the perpendicular or **sūcī** perpendicular, so to say. In this way, the learned obtained the elements of a figure by ratio (and proportion)."

In Fig.7, let AD and BC, when produced, meet at M. Let AR and BS be perpendicular to AB at A and B, respectively. Also, let AM and BM when produced, meet AR and BS at R and S, respectively. According to the rule,

1a. $BS = (DE.AB)/AE$; 1b. $AS = (AD.AB)/AE$

2a. $AR = (CF.AB)/BF$; 2b. $BR = (BC.AB)/BF$

3a. $MQ = (AR.BS)/(AR + BS)$;

4a. $AQ = (AR.AB)/(AR + BS)$; 4b. $BQ = (BS.AB)/AR + BS)$

5. $MQ = (AM.BS)/AS = (BM.AR)/BR$

Rational : Results $1a$, $1b$; $2a$, $2b$ and 5 follow from the similarity of triangles ABS, AED : RAB, CFB and ABS, AQM : BRA, MBQ, in order. From the similarity of triangles AQM, ABS and MQB, RAB, we have $AQ/MQ = AB/BS$, and $BQ/MQ = AB/AR$

or $AQ/BQ = AR/BS$

or $AQ/(BQ + AQ) = AR/(AR + BS)$. From this and a similar expression for BQ, results $4a$ and $4b$ follow. Also, substituting the value of AQ from $4a$ in $MQ/AQ = BS/AB$,

we get $MQ = \left(\frac{AR.AB}{AR+BS}\right)\frac{BS}{AB}$. This gives (3)

Ex.53 "Tell me here, the perpendicular from the root of a side in the form of a **vaṁśa** and the side resembling rajjus and along **sūcī** (sides) in front (of them and) meeting together (and) proceeding (up to the perpendicular)."

MADHYALAMBA

R.63b-64a "Subtract the face from the base. Divide (the remainder) by the base. The quotient multiplied by the **sūcī** perpendicular is the madhyalamba. Alternatively, twice the area divided by the sum of the base and the face, happens to be the **madhyalamba.**"

Let A be the area, b, the base, f, the face and s, the **sūcī** perpendicular of a quadrilateral. According to the rule, its madhyalamaba ($= m$) $= [(b - f)/b]$.s.

Alernatively, $m = 2A/(b + f)$

Madhyalamba plays the same role in finding the area of a cyclic quadrilateral, as the perpendicular in a triangle, provided the base is replaced by the sum of the base and the face in the case of quadrilateral.

Ex. 54a "In that very quadrilateral, tell the measure of the madhyalamba[77]."

While solving this problem Nārāyaṇa refers to Śrīdhara and Lalla, according to whom **madhyalamba** is half of the sum of perpendiculars, from the points of intersection of flank sides with the face, to the base. According to them, the **madhyalamba** of this quadrilateral ($= 378 + 448) / 2 = 413$. According to Nārāyaṇa, this is not true. For, according to this rule, area of this quadrilateral $= (½), (600 + 250) . 413 = 175525$, whereas actual area of the quadrilateral is 176400. This is obtained by subtracting the area of the triangle above the quadrilateral ($= 630000)/17$ from the area of the bigger **sūcī** triangle including the quadrilateral ($=3628800)/17$. Now, by Nārāyaṇa's R.63*b*-64*a*, the area of the quadrilateral also comes to 176400. Therefore, Nārāyaṇa's rule is correct and that given by Śrīdhara and Lala is not true.

GEOMETRIC SERIES FIGURE

R.64*b*-65*a* "Divide the face by the base. (Face and base etc.) multiplied by (the quotient) are the face etc. of the quadrilateral standing above that (quadrilateral). Divide the base by the face. Face, (base) etc. multiplied by (the quotient) happen to be the face etc. in the quadrilateral lying below (that quadrilateral)"

By this rule, from a given quadrilateral, a series of quadrilaterals can be formed which can be set one above the other and another series of quadrilaterals can be formed which can be set one below the other. In such quadrilaterals, lengths of their respective flank sides are a series in geometric progression and so are their bases, faces, respective diagonals, respective perpendiculars and areas.

If b be the base and f, the face of the given quadrilateral, the common ratio for the constituents of the former set (except the area) will be f/b and that for the latter set, b/f. The common ratio for the area will be (f^2/b^2) in the former case and (b^2/f^2) in the latter case.

Rational : Clearly, for the first set, since b is the base and f, the face of the given quadrilateral, the base of the next quadrilateral will be $(b.b/f=)f$, matching the face of the given quadrilateral. So, the next quadrilateral canbe set above the given quadrilateral, and so on the similarly for the other set. It is also clear that respective flank sides, respective perpendiculars etc. of the next quadrilateral, so obtained by the rule, will match those of the figure of the next quadrilateral and hence the rule.

Ex. 54b-c "O friend, tell the rational faces etc, of (similar) quadrilaterals lying below and standing above the quadrilateral (stated earlier)[78]."

CIRCLE

R. 65b-66 "subtract the square-root of the difference of squares of the diameter and the chord from the diameter. (half of) the difference happens to be the height of the chord. Subtract the height of the chord from the diameter. Multiply the difference by the height of the chord. Twice the square-root of the product is the chord.

Divide the square of the chord by 4 times (its) height. (The quotient) added to the height is the diameter[79]."

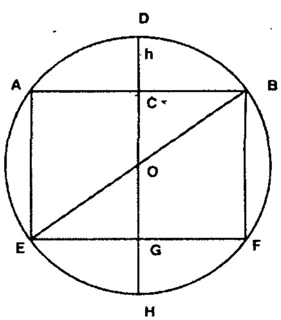


Fig-8

In Fig.8, let AB (= c) be the chord, CD (= h), the height of the chord AB and O, the centre of the circle Let BO be joined and produced to meet the circumference of the circle in E. Also, let the chord EF be drawn perpendicular to the diameter DH (= h) and let BF be joined. According to the rule,

(1) $h = [d - (d^2 - c^2)^{1/2}]/2$

(2) $c = 2\,[(d - h)h]^{1/2}$

(3) $d = (c^2/4h) + h$

These rules have been stated by Umāsvāti (c. 150 B.C. or A.D.) in his gloss on his own treatise **Tattvārthādhigama-Sūtra**[80].

Ex. 55 "In a circle, the diameter is 10 (and) the chord measures 8. (Tell) me the measure of the height of that (chord). (Tell) the chord from ' the diameter and the height' and the diameter from ' the chord and the height'."

RELATION BETWEEN TWO CHORDS AND THEIR HEIGHTS

R.67. "Twice the difference of heights of two chords is equal to the difference of the two chords conformable to the side and the upright of (a rectangle within a circle). The area (of the rectangle) is equal to their product. Sum of the squares (of the chords) should be obtained by the method stated earlier."

In Fig.8, let the lenghts of the two chords forming an inscribed rectangle of area A be x and y, $x > y$ and heights of these chords be h and k, respectively. According to the rule, $xy = A$ and $x - y = 2\,(k - h)$. Now, separate values of x and y can be obtained by R. 35 or R. 36. Let ABFE be a rectangle inscribed within the circle (Fig.8) and let the diameter DH of the circle meet the chords AB and EF, perpendicularly at C and G. Also, let $AB\ (= EF) = x$ and $AE\ (= FB) = y$ and heights of AB and AE be h and k, respectively, Suppose, $x > y$. If d be the diameter of the circle, we have,

$d\,(= DH) = DC + CG + GH = y + 2h$. Similarly $d = x + 2k$. Now, the rule can be easily derived.

Ex. 56 "The measure of the area of a rectangle inscribed in a circle is 480 and the measures of the heights (of the chords forming the rectangle) is 9 (and) 2. Tell quickly (the measures of) the diameter and the chords (of the circle)."

HEIGHTS OF COMMON CHORD

R.68. "Subtract the portion of the line, joining the centres of two interesecting circles (and) intercepted between them from their diameters (severally). Add the difference (and also) multiply them (severally) by the intercepted portion. The greater and the smaller products, divided by the sum, happen to be the heights (of the common chord) in the smaller and the greater circles, respectively.[81]"

Let two circles interesect at P and Q and let $ABCDE$ be the line through their centres. According to the rule,

$$BC = \frac{BD\,(AD - BD)}{(AD - BD) + (BE - BD)}.$$

and $$CD = \frac{BD\,(BE - BD)}{(AD - BD) + (BE - BD)}.$$

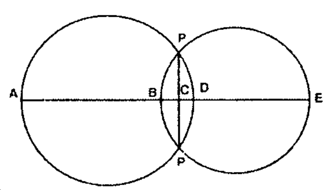


Fig.9

Rational : We have, $AC \times CD = PC^2 = BC \times CE$

or $(AD - BD + BC)\,(BD - BC) = BC\,(BE - BC)$

and $(AD - CD)\,CD = (BD - CD)\,(BE - BD + CD)$

or $BD.AD - BD^2 = BC.AD - BC.BD + BC.BE - BC.BD$

and $BD.BE - BD^2 = AD.CD - BD.CD + BE.CD - CD.BD$

From these two relations, the results are easily derivable.

Ex. 57. "Two circles of diameters 13 and 20 meet (like) the darkness covering the moon. The measure of portion of the line joining their centres and intercepted between them happens to be 6. O leanred, tell quickly the heights of the arcs and the measure of the common chord."

TRIGONOMETRY

APPROXIMATE EXPRESSION FOR SINE FUNCTION

R. 69. "Subtract the arc from half of the circumference. Square (the difference). That (i.e., the square), subtracted from the square of half of the circumference (and also), added to the square of the circumference are called **Guṇya** and **Hāra,** respectively, **Guṇya** multiplied by the diameter (and then) divided by one-fourth of **hāra** happens to be the chord. The first **jyā** subtracted from the near **jyā** are parts of the diameter.[82]

Let an arc AP (= s) subtend an angle θ at the centre of a circle of radius R (= OP) and diameter D. Let PM be perpendicular to OA at M. According to the Hindus, **Jyā** s (=$R \sin \theta$) = PM, **Kojyā** s (= $R \cos \theta$) = OM and **Utkramajyā** s (= $R - R \cos \theta$) = MA.

Let the circumference of the circle be C. According to the rule, **jyā** of an arc s will be given by **jyā** s = **guṇya.4D** / **hāra** where

guṇya = $\left(\frac{C}{2}\right)^2 - \left(\frac{C}{2} - s\right)^2$

hāra = $C^2 + \left(\frac{C}{2} - s\right)^2$

Thus, chord (= $2R \sin\theta$) = $4D \dfrac{\left[\left(\frac{C}{2}\right)^2 - \left(\frac{C}{2} - 2\theta\right)^2\right]}{\left[C^2 + \left(\frac{C}{2} - 2\theta\right)^2\right]}$

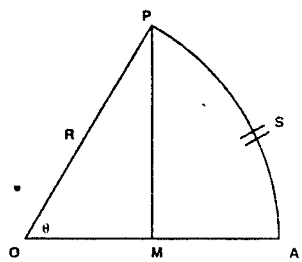


Fig. 10

This on simplification gives (θ being in degrees),

$$\sin \theta = \frac{4\theta(180 - \theta)}{40500 - \theta.(180 - \theta)} \quad \ldots(1)$$

ALTERNATIVE RULE

R.70 "Subtract the arc from the circumference. Multiply the difference by the arc. Keep (the product) at two places. Multiply (the product at one place) by the diameter (and then) divide by the difference of 'one-fourth (of the product) at the other (place) subtracted from 5 times the square of one-fourth of the circumference. (the quotient) happens to be the chord and this (rule) is of use in astronomy[83]."

With the above notatons, according to the rule,

$$\text{chord} = \frac{(c - 2\theta)\, 2\theta . 2R}{5(c/4)^2 - \frac{1}{4}.2\theta(c - 2\theta)}$$

This on simplification gives the formula (1) given above. The rule was first given by Bhaskara I (before 629 A.D.)[84]. From his statement, it seems that the formula was knows to Āryabhaṭa I.[85] Gupta has discussed the rule in detail.[86]

Ex. 58 "O learned, tell quickly the (corresponding) chords whose arcs are 50 multiplied by 1 to9, in the circle whose diameter is 300 and circumference 900."

TO FIND ARC FROM THE CHORD

R. 71 "Divide 4 times the diameter by the chord (and then) multiply by **ādya** (i.e., the first). Add **ādya** (to the product). Multiply 5 times the square of the circumference by **ādya.** Divide the product (by the above sum). Subtract (the quotient) from the square of thecircumference. Subtract (the square root) of this difference from the circumference. Half of (the result) happens to be the arc."

Let J be the chord corresponding to the arc s of a circle of diameter D and circumference C. According to the rule,

$$s = \left[C - \left\{C^2 - (5C^2.\text{ādya}) / \left(\text{ādya} + \frac{4D}{J}.\text{ādya}\right)\right\}^{\frac{1}{2}}\right] / 2$$

where **ādya** stands for s $(c - b)$. The rule is converse of R.70.

SIDE OF AN INSCRIBED REGULAR POLYGON

R.72. "The chord of the arc (obtained by) dividing the circumference (of the circle) by the number of sides (of the inscribed figure) or, the chord (obtained by) taking the measure of the circumference equal to the number of sides (and that of) the arc equal to unity (and then transformed with respect to the desired circle) is the measure of a side (of the inscribed regular) triangle etc. (in the circle) [87]."

Ex. 59. "O learned, tell me separately the mesure of a side of regular triangle etc. inscribed in a circle of diameter 1000."

SERIES-FIGURE

R.73-76a "Take the first term less half of the common difference as the face, the face added to (the product of) the common difference multiplied by the number of terms as the base and the number of terms as the perpendicular, (then) the area of (such a figure) happens to be equal to the sum of the series in A.P. Multiply the common defference by the perpendicular of the part. (The product) added to (its) own face is the base of that part. In the case of a negative face, the sides approach (each other and then after) crossing each oter, (the face) increases uniformly. The base of the lower triangle happens to be positive and that of the upper, (which is its face), negative. Divide the base (and) the face, severally by ' the base less the face.' (The quotients) multiplied by the perpendicular happen to be the perpendiculars of the triangles (and) the difference of their areas is equal to the sum of the series (in A.P.)[88]."

The series-figures considered here are plane figures resembling a trapezium (or a triangle) with equal flank sides.

Let ***a*** be the first term of a series in A.P., ***d***, its common difference and *n*, number of terms. Also, let ***b*** be the base of the corresponding series figure, *f*, its face, *P*, its perpendicular (from a point on its face) to its base and *p*, its perpendicular (from a point on the face of a part) to the base of that part. The rule states that,

(1) If a series-figures be formed such that face = $a - d/2$, its base = $a + nd-d/2$ and its perpendicular, $P = n$, then its area = $(n/2)\,[2a + (n-1)d]$. = sum of a series in A.P.

(2) The base of a part of the series figure = $f + pd$, p being the perpendicular distance between the parallel sides of that part.

(3) In case the series figure is made of two triangles, the base is considered as positive and the face, negative.

(4) If the series figure be made of two triangles,

(a) the perpendicular to the base = $P.b/(b-f)$

(b) the perpendicular to the face = $P.f/(b-f)$

(c) the area of the series figure = difference between the areas of the two triangles. = sum of the series in A.P.

According to Śrīdhara, if the series figure consists of two triangles, the base is negative and the face positive.[89]

Ex.60 "A series figure (is formed with) 1 as the first term and 1 as the common difference, the number of terms being5. O learned, tell me the base, the face and the bases of the parts (if) the perpendicular (in each part) be.1"

Ex. 61 "A series figure (is formed with) 1 as the first term and 1 as the common difference, the number of terms being 3½. O friend, tell the face etc. of the series figure."

Ex. 62. "O friend, tell quickly (the face etc. of the series figure in which) 3 is the first term, 2, the common difference and 5, the number of terms. (Tell) quickly the face etc. of the series-figure in which) ½ is the first term, 3, the common difference and $3\text{-}\frac{1}{3}$, the number of terms."

Ex. 63 "O friend, a series figure (is formed with) 3 as the first term, 7 as the common difference and 1/7 as the number of terms. O mathematician, if you know, tell how the series figure looks."

Ex. 64. "If you know, tell, how the series figure looks (if) the first terms (of the series in A.P.) be 1, the common difference, 1 and the number of terms, -5."

Ex. 65 "O friend, tell here quickly the series figure, (if) the first term be 25, the measure of the common difference be 5 (and) the number of terms, -9."

CONVERSE RULE

R.76b-77 "Subtract the face from the base. The difference divided by the perpendicular (is) the common difference, the face added to half of the common difference (is) the first term (and) the perpendicular is the number of terms of a series (in A.P.) whose sum is equal to the area of (the given) series figure.[90]

In the case of a negative face the madhyalamba should be taken as the perpendicular. (However) in such a case if the quadrilateral be a scalene quadrilateral, the first, term and the common difference, so obtained, are not correct."

Let the area of a series figure be equal to the sum of a series in A.P. With the notations of R.73-76b, the rule states that,

(1) $a = f + d/2$

(2) $d = (b - f)/P$ and

(3) $n = P$. However, in case of a negative face, P should be replaced by the **madhyalamba** and in such a case if the quadrilateral be a scalene one, the results (1) and (2) are not obtained, correctly. The rule is the converse of R. 73-76*a*.

Ex.66 "Tell separately, the first term, the common difference and the number of terms of the series (in A.P.) for the equibilateral etc. quadrilaterals stated earlier in the regular method.[91]"

Ex. 67. "The flank sides (of a series-figure) are 15 (and) 20, the face is -5 (and) the base, 30. Tell the first term, the common difference and the number of terms (there of)."

Ex. 68. "O mathematician, tell the first term, the common difference, the number of term (and) the area (of the series figure), where the face is -8, the base, 8 and the **madhyalamba**, 6."

Until otherwise stated the following notations have been used throughout sections dealing with R. 78 to R.130. Symbol a, stands for the side, b, the upright and d (or h), the diagonal of a rectangle (or hypotenuse of a right angled triangle). Symbols k, m, and n stand for optional numbers and p, q, r, and s, for seeds. Here, it should be noted that the Hindus treated a right angled triangle as part of a rectangle[92].

FORMATION OF RATIONAL AND INTEGRAL TRIANGLES

R.78. "Square of the side happens to be equal to the difference of squares of the diagonal and the upright. The diagonal and the upright should be obtained by supposing an optional number to be their difference."

That is, $d^2 - b^2 = a^2$ and $d - b = n$. From these equations d and b can be separately obtained by R. 32 and R. 31 (chapter.I).

Ex. 69 "If you have laboured hard in mathematics, tell the diagonal and the upright of a rectangle whose side is 24, in many ways."

ALTERNATIVE RULES

R. 79. "Multiply twice the side by an optional number. Divide (the product) by the square of the optional number less 1. (The quotient) is the upright. Multiply that (i.e., the quotient) by the optional number. (The product) lessened by the side is the digonal.[93]"

That is, $b = 2an/(n^2 - 1)$ and $d = 2an^2/(n^2 - 1) - a$

R. 80. "Add 1 to the square of an optional number. Divide twice the diagonal by the sum. (The quotient) subtracted from the diagonal, is the upright and that (i.e., the quotient) multiplied by the optional number is the base.[94]"

That is, $a = 2dn/(n^2 + 1)$ and $b = d - 2d/(n^2 + 1)$

Ex. 70 "O learned, if you know, tell in many ways, the upright and the side of the figure whose diagonal is 650."

R.81 "Multiply twice the hypotenuse by an optional number. Divide the product by ' the square of the optional number plus 1'. (The quotient) is the upright Multiply (the quotient) by the optional number. ('The product) less the hypotenuse' is the side[95] "

That is $b = 2hn/(n^2 + 1)$ and $a = 2hn^2/(n^2 + 1) - h$

FORMATION OF RECTANGLES (OR RIGHT ANGLED TRIANGLES FROM SEEDS)

R. 82. "The two numbers used in forming a rectangle (or a right angled triangle) are called (**Bījas** (i.e., Seeds), The sum and the difference of their squares are the diagonal (or the hypotenuse) and the upright (respectively and) twice their product is the side.[96]"

That is, $a = 2rs$, $b = r^2 - s^2$, and d (or h) $= r^2 + s^2$.

Ex. 71 "O learned if you know, tell in many ways, the base, the upright and the diagonal, (these being) rationals, of arectangle (and) tell (their) measures in

other set (of rectangles)."

ALTERNATIVE RULE

R. 83. "The product of the sum and the difference of the seeds is the upright and the two quantities obtained from the squares of those (i.e., from the squares of the sum and the difference) by the method of concurrence are the diagonal and the side of a rectangle.[97]"

That is, $b = (r + s)(r - s)$,

$d = [(r + s)^2 + (r - s)^2]/2$

and[98] $a = [(r + s)^2 - (r - s)^2]/2$

FORMATION OF SEEDS

R. 84. "The upright is divided by an optional number. (Quantities) obtained by the method of concurrence from the denominator and the quotient are the seeds. Half of the base is divided by an optional number. The denominator and the quotient are the seeds."

That is, $r = \frac{1}{2}\left(\frac{b}{n} + n\right)$ and $s = \frac{1}{2}\left(\frac{b}{n} - n\right)$.

Alternatively, $r = (a/2n)$, and $s = n$

Ex. 72 "O friend tell the side and the diagonal where the upright is 12. What are the upright and the diagonal where the side is 12."

ALTERNATIVE RULE

R. 85 "Divide the square of the product of the two seeds by an optional number. Square-root of the quotient and the divisor happen to be the other pair of seeds. If the roots are not rational, the seeds are so."

That is, $r = \sqrt{(p^2q^2)/n}$ and $s = \sqrt{n}$, p, q being a given pair of seeds.

FORMATION OF RATIONAL AND INTEGRAL QUADRILATERALS.

FORMATION OF RATIONAL AND INTEGRAL EQUIBILIATERAL TRAPEZIUMS

R.86-87 "Two desired rectangles having equal sides (but) unequal uprights and, diagonals are formed (from the seeds). The sum and the difference of their uprights are the base and the face (of a trapezium, respectively). The small diagonal is (each of) the two flank sides and the greater diagonal (each of) the two diagonals. The two sides are the two perpendiculars. The two uprights are the **sandhi** (i.e., the link) and the **pīṭha** (i.e., the complement of the segment), respectively. The product of the diagonals divided by the side is the (circum) diameter (and) the area (of the greater rectangle) is the area (of the trapezium, so formed)[99]."

Let one pair of seeds be r and s. Then another set of seeds say, p and q, can be obtained by R.85 From these set of seeds two rectangles can be formed in which the greater sides have the same lenghts $2pq = 2rs$, as $pq = rs$. By suitably juxtaposing these retangles, a equibilateral trapezium can be formed by the rule.

Let $(p^2 + q^2) > (r^2 + s^2)$, and $(p^2 - q^2) > (r^2 - s^2)$.

According to the rule an equibilateral trapezium can be formed such that its base = $(p^2 - q^2) + (r^2 - s^2)$, face = $(p^2 - q^2) - (r^2 - s^2)$, each flank side = $r^2 + s^2$, each diagonal = $p^2 + q^2$, each perpendicular = $2pq = 2rs$, **sandhi** (link) = $r^2 - s^2$, **pīṭha** (complement of the segment) = $p^2 - q^2$, circumdiamenter = $\frac{(p^2 + q^2)(r^2 + s^2)}{2pq}$ and area = $2pq\,(p^2 - q^2)$. obviously, the sides, the diagonals, the altitude, the segments of the base made by the perpendiculars and the area of such a trapezium will all be rational, if the seeds are rational and these all will be integral, if the seeds are so.

Ex.73. "O friend, tell quickly, the base, the face, the flank sides, the diagonal and the perpendicular etc., all these being rationals, of the equibilateral trapezium."

FORMATION OF DIFFERENT TYPES OF TRAPEZIUMS

R. 88-90 "The sum and the difference of the diagonal with the side and with the upright, are separately multiplied by the side (and) by the upright, (in order). Rectangles formed from the original seeds are called the first born, (those, from) the square-roots (of the products of the side), the side- (born and those, from) the square -roots (of the products of upright), the upright-(born).

Quadrangle formed from the first-born (and) the side- (born) is equitrilateral, from the first-born (and) the upright - (born) is either equi-trilaleral or has (its) base equal to (its) diagonal (and), from the side-born and the upright-(born) has (its) base equal to (its circum) diameter. The flank sides, the diagonals etc. (of the quadrangle) should be obtained by the method of equibilateral quadrangle (stated earlier)."

Rational : Let r and s be the seeds. Let the first rectangle formed from these have its side and its upright equal to r and s respectively, so that its diagonal will be equal to $(r^2 + s^2)^{1/2}$. Therefore, the seeds of the side-born rectangle will be $[r\{r^2 + s^2)^{1/2} + r\}]^{1/2}$ and r $(r^2 + s^2)^{1/2} - r\}]^{1/2}$ and those of the upright - born $[s\{(r^2 + s^2)^{1/2} + s\}]^{1/2}$ and $[s\ \{(r^2 + s^2)^{1/2} - s\}]^{1/2}$. Now, the side, the upright and the diagonal of the first-born rectangle will be, $2r^2$, $r^2 - s^2$ and $r^2 + s^2$, those of the side-born, $2rs$, $2r^2$ and $2r(r^2 + s^2)^{1/2}$ and those of the upright-born, $2rs$, $2s^2$ and $2s(r^2 + s^2)^{1/2}$, in order.

Trapezium formed from the first-born and the side-born will always be equitrilateral, for its face

$$= 2r^2 - (r^2 - s^2) \text{ by R.86 – 87, } = r^2 + s^2$$

= small diagonal between the two rectangles

= (equal) flank side of the trapezium by R88-90.

Trapezium formed from the first-born and the upright- born may be equitrilateral, for its base

$= r^2 - s^2 + 2s^2$ by R.86-87 $= r^2 + s^2$

= smaller diagonal between the two rectangles

= (equal) flank side of the trapeziumby R.88-90.

If however, $r^2 + s^2 > 2s\,(r^2 + s^2)^{1/2}$, the above base,

= greater diagonal between the two rectangles by R.86-87.

= (equal) diagonal of the trapezium by R. 88-90.

Trapezium formed from the side-born and the upright-born will have its circumdiameter,

= product of diagonals of the two rectangles divided by the base.

$$= \frac{2r\,(r^2+s^2)^{1/2}\,.\,2s\,(r^2+s^2)^{1/2}}{2rs} = 2(r^2+s^2).$$

= sum of the uprights of the two rectangles

= base of the trapezium by R. 86-87.

Ex. 74 "O mathematician, if you are well versed (in mathematics), tell the equitrilateral quadrangle or (the quadrangle whose) base is either equal to its diagonal or, to its (circum-) diameter."

FORMATION OF RATIONAL EQUITRILATERAL TRAPEZIUMS : ALTERNATIVE RULE

R. 91-93a "The square of the diagonal of a rectangle is the side, the difference of the sqaures of the side and the upright is the segment, twice their product is the perpendicular, twice the greater between the side and the upright, multiplied by the diagonal is the diagonal, the cube of the diagonal divided by the smaller (between the side and the upright) is the (circum-) diameter and the cube of the greater multiplied by 4 times the smaller is the area, (which is equal to the product of) 'half of the sum of the base and the face' multiplied by the perpendicular, of the equitrilateral trapezium[100]."

Rational : Let a rectangle be formed such that its side, upright and the diagonal are r, s, and $(r^2 + s^2)^{1/2}$, respectively. From these an equitrilateral trapezium can be formed such that its flank side (and also the face), segment of its base made by the perpendicular and its perpendicular are $r^2 + s^2$, $r^2 - s^2$ and $2rs$, respectively, Also, if $r > s$, its diagonal, circum-diameter, and area are $2r\,(r^2 + s^2)^{1/2}$, $\frac{(r^2+s^2)^{3/2}}{s}$, and $4r^3s$, respectively. Clearly, area of the trapezium

= (1/2) (base + face). perpendicular

$$= \frac{r^2 + s^2 + 2(r^2 - s^2) + r^2 + s^2}{2} \cdot (2rs) = 4r^3s.$$

FORMATION OF CYCLIC, RATIONAL OR INTEGRAL SCALENE QUADRILATERAL.

R. 93b-97a. "The upright and the side of the bigger rectangle (among two given or assumed rectangles) multiplied by the diagonal of the smaller (rectangle, separately), are the face and the base. The upright and the side of the smaller (rectangle) multiplied by the diagonal of the bigger rectangle are the two flank sides. All sides multiplied by diagonal of the smaller (rectangle) are all the sides of a scalene quadrilateral.

The sum of the products of the sides and (the product) of the uprights is the **guṇa**. The sum of the products of the side (of one rectangle) and the upright of the other is the other **guṇa**. The **guṇas** (separately) multiplied by the diagonal of the smaller (rectangle) are the diagonals of the quadrilateral. The diagonal of the smaller (rectangle) multiplied (the product of) the diagonals (of the 2 rectangles) is the (circum) diameter and half of the product of the diagonals of the guadrilateral is the exact area of the quadrilateral.

[The smaller and the greater **guṇas** multiplied separately by the greater and the smaller sides of the smaller rectangle, respectively, are the perpendiculars and those multiplied separately, by the smaller and the greater sides are the **pīthas** (i.e., complements of the segments). **Pīthas,** subtracted from the base, are the **sandhis** (i.e., links[101].]

Part of the rule within square backets is missing from the text. It has been reconstructed with help of the solution of the problem, given by Nārāyaṇa, after the rule.

The rule gives a method for the formation of scalene quadrilaterals inscribable within circles, whose sides, diagonals, perpendiculars segments (of sides and diagonals made by perpendiculars from the vertices as also of diagonals made by the otherdiagonal), areas and circumdiameters are all rationals or integrals, as desired.

First of all 2 rectangles are formed (R.82). Multiplying the sides of each rectangle by the base of the other and again, by the upright of the other, 4 rectangles (or right angled triangles, so to say) are obtained. By the judicious juxtaposition of these 4 right angled triangles, a scalene quadrilateral is formed which has the properties as desired. For integral scalen quadrilateral, integral values are taken.

Let $(r^2 - s^2)$, $2rs$, and $(r^2 + s^2)$ be the base, the upright and the diagonal of the smaller rectangle and $(p^2 - q^2)$, $2pq$, and $(p^2 + q^2)$ be those of the bigger rectangle, in order. According to the rule, a scalene quadrilateral can be formed such that its.

face $= (r^2 + s^2)^2 (p^2 - q^2)$; base $= 2pq(r^2 + s^2)^2$;

flank sides $= (r^2 + s^2)(p^2 + q^2)(r^2 - s^2)$, and $2rs(r^2 + s^2)(p^2 + q^2)$;

diagonals $= (r^2 + s^2).\ [4pqrs + (r^2 - s^2)\ (p^2 - q^2)]$,

and $(r^2 + s^2)\ .\ [2pq\ (r^2 - s^2) + 2rs\ (p^2 - q^2)]$;

perpendiculars $= (r^2 - s^2).\ [2pq\ (r^2 - s^2) + 2rs(p^2 - q^2)]$.

and $2rs\ [(p^2 - q^2)\ (r^2 - s^2) + 4pqrs]$;

piṭhas (complements of the segment)

$= (2rs)\ [2pq(r^2 - s^2) + 2rs(p^2 - q^2)]$,

and $(r^2 - s^2)\ [(r^2 - s^2)\ (p^2 - q^2) + 4pqrs]$,

sandhis (links) $= 2pq\ (r^2 + s^2)^2 - [2pq(r^2 - s^2) + 2rs\ (p^2 - q^2)\ (2rs)$,

and $2pq\ (r^2 + s^2)^2 - [(r^2 - s^2)\ (p^2 - q^2) + 4pqrs]\ (r^2 - s^2)$;

circumdiameter $= (r^2 + s^2)^2\ (p^2 + q^2)$;

area $= \frac{1}{2}\ [2pq\ (r^2 - s^2) + 2rs\ (p^2 - q^2)].\ [(r^2 - s^2)\ (p^2 - q^2) + 4pqrs)]\ (r^2 + s^2)^2$.

In fact there will be varieties of quadrilaterals according as the sequence in which the quantities $(r^4 - s^4)\ (p^2 + q^2)$, $(p^2 - q^2)\ (r^2 + s^2)^2$, $2rs(p^2 + q^2)\ (r^2 + s^2)$, and $2pq\ (r^2 + s^2)^2$ are taken : (1) The variety in which the two diagonals intersect at right angles (2) The other variety is of two kinds. In these the two diagonals intersect obliquely.

The quadrilateral described above is a quadrilaterl of the first variety. For the oblique varities the sides are in the following order

(a) $2pq(r^2 + s^2)^2$, $(r^2 + s^2)^2\ (p^2 - q^2)$, $(r^2 + s^2)$, $(r^2 - s^2)\ (p^2 + q^2)$ and $2rs\ (r^2 + s^2)\ (p^2 + q^2)$

and (b) $2pq(r^2 + s^2)^2$, $(r^2 + s^2)\ (r^2 + s^2)^2.\ (p^2 - q^2)$, $2rs\ (r^2 + s^2)\ (p^2 + q^2)$; and $(r^2 + s^2)\ (r^2 - s^2)\ (p^2 + q^2)$; Their diagonals will be (a) $(r^2 + s^2)\ [4pqrs + (r^2 - s^2)\ .\ (p^2 - q^2)]$, and $(r^2 + s^2)^2.\ (p^2 + q^2)$; and (b) $(r^2 + s^2)\ [2pq\ (r^2 - s^2) + 2rs\ (p^2 - q^2)]$, and $(r^2 + s^2)^2\ (p^2 + q^2)$, in order.

The circumdiameter will be the same in all the cases.

Ex. 75 "If you have pride in mathematics, tell me quickly, the base and the face etc. of the rational scalene quadrilateal."

FORMATION OF RATIONAL ISOSCELES OR SCALENE TRIANGLE.

R.97*b*-98*a* "Square of the perpendicular, kept at two places is (severally) divided by two optional numbers. (The respective) optional numbers are added to and subtracted from (the quotients). Halves of the results are the sides and the segments of the base of the triangle.[102]"

Let m, n be two optional numbers and p be the length of the perpendicular

from the vertex to the base of a triangle. According to the rule, $\frac{1}{2}\left(\frac{p^2}{m}+m\right)$ and $\frac{1}{2}\left(\frac{p^2}{n}+n\right)$ are the sides and $\frac{1}{2}\left(\frac{p^2}{m}-m\right)$ and $\frac{1}{2}\left(\frac{p^2}{n}-n\right)$ are the segments of the base made by the perpendicular from the vertex of the desired triangle. The triangle is formed by the juxtaposition of two rectangles (or right angled triangles) having a common upright.

In Europe, the rule is found to have been employed first by Bachet (c.1621).[103] In India, Brahmagupta (628 A.D.) gave the rule first.

Ex.76 "O learned, if you know, tell me the rational sides, (segments of) the base ad perpendicular of the isosceles and scalene triangles."

FORMATION OF EQUIBILATERAL TRAPEZIUM FROM A SCALENE TRIANGLE.

R. 98b-100 "The smaller side, the greater side, the perpendicular (from the vertex and) the base of the scalene triangle are the two (equal) flank sides, the two (equal) diagonals, the altitude and the base (of the trapezium, in order). Subtract the product of (opposite) flank sides from the product of the diagonals. (The difference) divided by the base is the face. The sum of the products of opposite sides is equal to the product of diagonals. The face etc. should be obtained from the knowns by the inverse process."

The rule gives a method for the formation of equibilateral trapezium. With the notations of R.97b, let a rational scalene triangle be formed, such that, $\frac{1}{2}\left(\frac{p^2}{m}+m\right)>\frac{1}{2}\left(\frac{p^2}{n}+n\right)$. According to the rule, an equibilateral trapezium can be formed such that each of its flank sides is $\frac{1}{2}\left(\frac{p^2}{n}+n\right)$: base, $\frac{1}{2}\left(\frac{p^2}{m}-m\right)+\left(\frac{p^2}{n}-n\right)$;

$\dfrac{\frac{1}{2}\left(\frac{p^2}{m}+m\right)^2-\frac{1}{2}\left(\frac{p^2}{n}+n\right)^2}{\frac{1}{2}\left(\frac{p^2}{m}-m\right)^2+\frac{1}{2}\left(\frac{p^2}{n}-n\right)}$; each of its diagonals, $\frac{1}{2}\left(\frac{p^2}{m}+m\right)$; and altitude, p. The trapezium is formed by the juxtaposition of two rectangles. It is easy to see that the sum of the products of opposite sides is equal to the product of diagonals.

Ex.77 "The sides of a triangle are 52 (and) 60, perpendicular (from the vertex) is 48 and the base, 56. Tell quickly the quadrangle."

FORMATION OF RATIONAL CYCLIC, SCALENE QUADRILATERAL.

R.101 "Product of the segments divided by the perpedicular happens to be the perpendicular touching the circle (and) lying below. The segments multiplied

by the interchanged sides (and) divided by the perpendicular, happen to be the flank sides of the (quadrilateral lying below)."

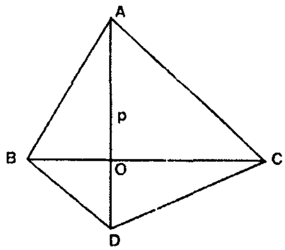


Fig. 11

With the notations of R.97b-100, let a rational scalene triangle be formed according to that rule, such that (Fig. 11) $AB = \frac{1}{2}\left(\frac{p^2}{m} + m\right)$ and $AC = \frac{1}{2}\left(\frac{p^2}{n} + n\right)$, perpendicular from the verlex A (= AO) = p, Clearly segments of the base BC made by the perpendicular AO, i.e.,

$BO = \frac{1}{2}\left(\frac{p^2}{m} - m\right)$ and $OC = \frac{1}{2}\left(\frac{p^2}{n} - n\right)$. Let AO be increased upto D, such that,

$$OD = (BO.OC)/AO = \frac{1}{4p}\left(\frac{p^2}{m} - m\right)\left(\frac{p^2}{n} - n\right) \qquad \ldots(1)$$

According to the rule, a rational cyclic scalene quadrilateral $ABCD$ can be formed,

such that,
$$BD = \frac{1}{4p}\left(\frac{p^2}{n} + n\right)\left(\frac{p^2}{m} - m\right) \qquad \ldots(2)$$

and
$$DC = \frac{1}{4p}\left(\frac{p^2}{m} + m\right).\left(\frac{p^2}{n} - n\right) \qquad \ldots(3)$$

Rational : Since, $AO.OD = BO.OC$ from (1), so the points A,B,C and D are concyclic and therefore $ABCD$ is a cyclic quadrilateral. Now from the similarity of triangles BOD and AOC, we have, $BD = (BO.AC)/AO$. Similarly, from the similarity of triangles ABO and CDO, $DC = (CO.AB)/AO$. Also, if we substitute the values of AC,AB,BO, CO and AO in terms of p,m, and n, *in* the expressions for BD and DC, We get the values of BD and DC as given by (2) and (3).

FORMATION OF PAIR OF RATIONAL QUADRANGLES

R.102-105 "The side whose uppermost and lowermost points meet a circle is known as Śiñjinī (a chord of that circle). The sum and the difference of the segments of the same diagonal (of an inscribed scalene quadrilateral are the base of one and the upright of another (pair of rectangles) having the same diagonal. If the greater (sum) is in one direction (i.e., the side), the smaller difference is in another (i.e., the upright of a rectangle). Opposite sides of the scalene quadrilateral are the side (and) the upright (and) from (another pair of rectangles (having the same diagonal). The diagonals of (all the four) rectangles are equal (circum) diameter of the inscribed quadrilateral."

The rule gives a method for the formation of a double pair of rectangles whose diagonals are equal and equal to the circumdiameter of a scalene quadrilateal. With the notations of R.97, let a rational, cyclic, scalene quadrilateral $ABCD$ be formed according to R.101 (for figure see R.101) such that its diagonals AD and BC intersect at right angles, say at O and sides, $AB = \frac{1}{2}\left(\frac{p^2}{m} + m\right)$, $AC = \frac{1}{2}\left(\frac{p^2}{n} + n\right)$

$$BD = \frac{1}{4p}\left(\frac{p^2}{n} + n\right)\left(\frac{p^2}{m} - m\right),$$

$$DC = \frac{1}{4p}\left(\frac{p^2}{m} + m\right)\left(\frac{p^2}{n} - n\right);$$ and segments of the diagonals,

$$BO = \frac{1}{2}\left(\frac{p^2}{m} - m\right)$$

$$OC = \frac{1}{2}\left(\frac{p^2}{n} - n\right), AO = p,$$

and $$OD = \frac{1}{4p}\left(\frac{p^2}{m} - m\right)\left(\frac{p^2}{n} - n\right)$$

Now, according to R. 102-105, let a double pair of rectangles be formed such that their sides are equal to AB, AC, AD and BC and uprights, DC,BD,$BO \sim OC$ and $AO \sim OD$, in order. Then the diagonals of such rectangles will be equal and equal to the circumdiameter of the cyclic scalene quadrilateal $ABCD$.

Rational : With given values of sides and uprights of rectangles as stated above, the diagonal of any rectangle $= \frac{1}{4p}\left(\frac{p^2}{m} + m\right)\left(\frac{p^2}{n} + n\right)$ = circumdiameter of the quadrilateal $ABCD$.

Ex. 78 "Tell the rectangles (having the same diagonal) and the scalene quadrilateral inscribed in a circle, and formed (from the triangle) stated earlier.[104]"

PAIŚĀCIKA MATHEMATICS

FORMATION OF PAIR OF RATIONAL EQUIBILATERAL TRAPEZIUMS HAVING THE SAME AREA

R.106-107 "The base and the upright (of an assumed rectangle) are the (respective) perpendiculars and the diagonal is the flank sides of the pair of (desired) equibilateral trapeziums. Multiply the given area by the square of an optional number. (The difference), divided separately by the (respective) perpendicular is their (respective) faces. These (faces), added (severally) to twice the other's perpendicular, are their (respective) base. (Finally) all the sides (and the perpendiculars) are to be divided by the optional number."

With the usual notations let A be the area of the desired trapeziums. The rule states that a pair of equibilateral trapeziums can be formed such that their bases are $\frac{n^2A+ab}{na}$, and $\frac{n^2A+ab}{nb}$; faces, $\frac{n^2A-ab}{na}$, and $\frac{n^2A-ab}{nb}$; altitudes, (a/n), and (b/n); and each flank side, (h/n). Clearly, both the trapeziums will have the same area, A.

Rational : One trapezium is formed by the juxtaposition of a rectangle with sides b/n and a/n and the other, with sides nA/b and a/n.

Ex. 79. "Friend, if you know, tell the face, the base, the flank sides and the perpendiculars of equibilateral trapeziums whose area is 10, in many ways."

FORMATION OF RATIONAL OR INTEGRAL QUADRILATEALS (CONTINUED)

R.108-109. "Divide the square of the area by the cube of an optional number. Add (the quotient) to the optional number. Half of the sum is the flank side. The side subtracted from twice the optional number is the face. The base is equal to the flank side. If the face be greater than the flank side that (face) is the base (and) the face is equal to the flank side. Divide the area by the optional number. (The quotient) happens to be the perpendicular in the equitrilateal trapezium.[105]"

Let A be the desired area and n an optional number. According to the rule, the desired equitrilateral trapezium will have its each flank side equal to $\frac{1}{2}\left(\frac{A^2}{n^3}+n\right)$, base, $\frac{1}{2}\left(\frac{A^2}{n^3}+n\right)$, face $2n-\frac{1}{2}\left(\frac{A^2}{n^3}+n\right)$, and altitude, $\frac{A}{n}$ In case the face becomes greater than the flank side, the face and the base are to be interchanged.

Rational : The desired trapezium is formed by the suitable juxtaposition of a rectangle wit sides $\frac{1}{2}\left(\frac{A^2}{n^3}-n\right)$ and $\frac{A}{n}$ and another rectangle with sides n and $\frac{A}{n}$. Clearly the area of the trapezium will be equal to A. Ex. 80 "O learned, if you know, tell quickly, the base, the face, the flank sides and the perpendicular etc. (all these being rationals), of the equitrilateral trapeziums, whose area is 12."

R.110 "4 optional numbers whose product is equal to the square of the area (are supposed). Half of their sum is kept at 4 places. These, (separately) subtracted from them, are the sides of the scalene quadrilateral.[106]"

Let A be the desired area of the quadrilateral. Now, let 4 numbers a,b,c and d be chosen such that $abcd = A^2$. Now, if $(a+b+c+d)/2 = s$, according to the rule, $s-a$, $s-b$, $s-c$ and $s-d$ are the sides of the desired scalene quadrilateral. Clearly, the area of such a quadrilateral will be A by R. (33b-34a) and so the rule is converse of that rule.

Ex. 81 "O learned mathematician, if you have proficiency in mathematics, tell quickly, the measures of the sides of the scalene quadrilateral whose area is 90, in many ways."

PROPERTIES OF TRIANGLES AND QUADRILATERALS (CONTINUED)

R.111. "Multiply the difference of the square of the base and the face by the area of that part. Divide (the product) by the whole area. Add the square of one's own face (i.e., face of the part) of the quotient. Square-root of the sum is the base of that part. The perpendicular (of the part) should be obtained by the method stated earlier.[107]"

The rule gives a method for finding the bases and the perpendiculars of several parts into which a equibilateral trapezium is divided by lines parallel to the base. Let b be the base, f, the face and a, the area of the trapezim and b', f' and a' be those of a part of it, in order. According to the rule, $b' = [(a'/a) \cdot (b^2 - f^2) + f'^2]^{1/2}$.

The perpendicular is to be obtained by R.34b. The rule is obtained by comparing similar figures.

Ex. 82 "Tell quickly the bases etc. of the parts (of the equibilateal trapezium) whose base is 20, face, 2, (each) flank side, 15 and the areas of the parts are 7, 32 and 93 (in order)."

FORMATION OF PAIR OF RATIONAL QUADRANGLES (CONTINUED)

R.112-113 "Multiplier of the perimeter and multiplier of the area, divided (separately) by the smaller multiplier (amongst them) is called **Phala** (and) **vṛti** (in order. Square of their product multiplied by an optional number happens to be the upright. **Vṛti** is multiplied by the optional number. (The product) less one is subtracted from that (i.e., from the upright). (the resulting difference) multiplied by (the former) difference determines the other side of the rectangle. The upright and the side of the the (pair's) other rectangle should be obtained from the product and the sum.[108]"

The rule gives a method for the formation of a pair of rectangle such that their perimeter are equal but the area of one is an integral multiple of that of the other or their areas are qual but the perimeter of one is an integral multiple of that of the other. Let x, y represent the base and the upright of one rectangle u, v, those of the pair, in order, such that,

$$2s\,(x + y) = 2t\,(u + v) \quad \text{.... (1)}$$

$$\text{and} \quad pxy = quv \quad \text{....(2)}$$

s, t, p and q being known integers. The rules gives the following solutions.

$$x = \left(\frac{nq}{p} - 1\right)\left[\frac{ns^2g^2}{t^2p^2} - \left(\frac{nq}{p} - 1\right)\right] \qquad ....(3)$$

and $$y = \frac{ns^2q^2}{t^2p^2} \qquad ....(4)$$

Putting these values in (1) and (2), We get the values of $u + v$ and uv from which u and v can be separately obtained by R.35b and R.31(chapter.1). Thus the desired pair of rectangles is obtained.

Ex. 83"In two rectangles, the area of the first is twice that of the other and their perimeters are equal or the perimeter of the first is twice that of the other and their area are equal. Tell (them.)"

Formation of Pair of Rational or Integral Triangles

R.114-115 "Cross-multiply the squares of the multipliers of the perimeters by the multipliers of the areas. Divide the greater produt by the smaller product. The quotient is separately multiplied by 6 and 2. The greater product and one less than smaller (product) are the seeds (for the formation of rectangles). (Also) their difference and twice the smaller (seed) are the (other pair of) seeds. Triangles should be formed from rectangles formed from the seeds by the method stated[109] earlier (such that) twice the side (of the rectangle) is the base (of the triangle).[110]"

The rule gives a method for the formation of pair of isosceles triangles, the perimeters and the areas of the same pair being related in given proportions. Let s and s' be the perimeters of two rational isosceles triangles whose areas are A and A', respectively, such that, $s{:}s' = m{:}n$ and $A{:}A' = p{:}q$, m,n,p and q being known integers. The rule stated that $(6a, 2a-1)$ and $(4a + 1, 4a - 2)$ are the seeds for the formation of 2 rectangles where a n^2p/m^2q or m^2q/n^2p according as $n^2p > m^2q$ or $m^2q > n^2p$. The rectangles are to be formed by R.82.

Now, according to the ruie, the dimensions of one triangle of the pair are given by, base = $24ma(2a-1)$, (each) flank side = $m.\ [(6a)^2 + (2a - 1)^2]$ and altitude = $m\ [(6a)^2 + (2a - 1)^2]$ and those of the other pair, by base = $4n\ (4a - 1)\ (4a - 2)$, (each) flank side = $n\ [(4a - 1)^2 + (4a - 2)^2]$;

and altitude = $n\ [(4a - 1)^2 - (4a - 2)^2]$

Here, it may be noted that by multiplying the above values by m^4q^2 or n^4p^2 as the case may be, we get a pair of isosceles triangles whose diamensions are integral.

Rational : With the values stated, the triangles formed are isosceles and it is easily seen that the ratio of their perimeters is m/n and that of the areas is p/q, as assumed.

Ex. 84 "Perimeters and areas of two isosceles triangles are equal. O friend, (if) you are a mathematicians (and) know, tell their sides etc."

The example was first given by Mahāvīra.[111]. Later on, the case was dealt with by European mathematicians, Frans van Schootan the Younger (657), J.H. Rahn (1697) and others.[112]

Ex. 85 "Two isosceles triangles have their perimeters equal and area of the first is twice that of the other and in another case (their) areas are equal and perimeter of the first is twice that of the other. (Tell) the seeds and their areas (etc.)."

This example was also first given by Mahāvīra.[113]

Ex. 86 "O learned, if you know mathematics, tell the altitudes, the sides etc. (of two isosceles triangles), if the area (of the second) is twice that of the first (and) and the perimeter (of the second) is observed to be thrice (that of the first)."

FORMATION OF RATIONAL OR INTEGRAL QUADRILATERALS (CONTINUED)

R. 116 "The square-root of the difference of squares of (the exact and the rough) areas, added to and subtracted from the square of an optional number, are the base and the face, the exact (area) the altitude and the rough (area) is equal to (each of) the flank sides in the equibilateral trapezium. In an equitrilateral trapezium, the (above) square-root, added to the rough area, is the square of the desired optional number. That (i.e., the square of the number) added to the square-root, is the base. [In both the cases, all sides, altitudes etc. are to be divided by the optional number.]"

Part of the rule inside the square brackets is missing in the text. This part of the rule has been reconstructed with help of the solution of the problem given by Nārāyaṇa. The rule gives a method for the formation of a rational equibilateral or an equitrilateral trapezium, the rough and the exact areas of the same being given. If R be the rough area and E, the exact area of an equibilateral trapezium and n, and optional number, according to the rule, a equibilateral trapezium can be formed such that its base $= [n^2 + (r^2 \backsim E^2)^{1/2}]/n$,

face $= [n^2 + (R^2 \backsim E^2)^{1/2}]/n$,

(each) flank side $= R/n$, and altitude $= E/n$.

In the case of an equitrilateal trapezium, n must be chosen, such that $n^2 = R + (R^2 \backsim E^2)^{1/2}$.

The trapezium is formed by justaposition of a rectangle with sides $(R^2 \backsim E^2)^{1/2}/n$ and E/n with, another, with sides n and E/n.

Rational : Naturally, rough area of such a trapezium will be R ($R.8b$) and exact area, $E(R.34b)$. If $n^2 = R + (R^2 \backsim E^2)^{1/2}$, the face of the trapezium will be equal to R/n = (each) flank side, and so the trapezium will be an equitrilateral one.

Ex. 87 "O friend, if you know, tell the face etc. of equibilateral and equitrilateral trapeziums, where the rough area is 104 and the exact area,96."

PROPERTIES OF TRIANGLES OR QUADRILATERALS (CONTINUED)

R.117 "If the side or the upright or the diagonal (of a desired rectangle) or their sum or their difference be equal to its area, (supposing) an optional number to be equal to that, (i.e., the side etc. of a suitable rectangle) divided by the area (of that suitable rectangle), (the side etc.) of the desired rectangle should be told by multiplying that (i.e., the side etc. of the suitable rectangle) by that optional number.[114]"

Ex. 88 "The side or the upright or the diagonal or the sum or the difference of any two or the perimeter of a rectangle is equal to its area. O **Ārya**, tell them quickly."

Ex.89 "One person is working on the perimeter (of a square) at the rate of the perimeter (of a square of side one **daṇḍa** in half a day). The other is working (on the same) at rate of half a **nivartana** (in a day). They do it in equal number of days. O friend, tell the sides. If the figure be a rectangle or a triangle, (them also, tell) the sides."

FORMATION OF RATIONAL OR INTEGRAL TRIANGLES (CONTINUED)

R.118 "Divide twice an optional number by 'the square of the optional number less 3'. Add 1 to thrice the square (of the quotient). Twice the square-root of the sum is the base. 1 added to and subtracted from (the base) are the flank sides."

The rule gives a method of forming integral (or rational) triangles whose sides differ by unity. Let n be an optioanl number. According to the rule, the base of the required triangle = $2x$, and the fland sides = $(2x \pm 1)$, where, $x = (3y^2 + 1)^{1/2}$, such that $y = 2n/(n^2 - 3)$

Rational : Let the desired rectangle ABED formed by the juxtaposition of two rectangles AFCD and FBEC with diagonals x-1 and x+1, respectively. obviously these diagonals will form the flank sides AC and BC of the triangle ABC, as shown in Fig. 12.

Let the sum of the bases of these rectangles be x and let these rectangles have a common upright, $FC = y$, Then, the separate bases of these rectangles will be (x/2--2) and (x/2 + 2), respectively (R.36). Now, from right angled triangle AFC, we get, $(x - 1)^2 - (x/2 - 2)^2 = y^2$. which leads to $\frac{3}{4}x^2 - 3 = y^2$ ...... (1)

Again, let $x = 2t$ and $y = 3m$. Thus, the equation reduces to $3m^2 + 1 = t^2$. Now, let $m = 2n/(n^2 - 3)$, where in is an optional number.

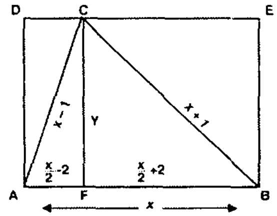


Fig. 12

This gives $t = \left[3\left(\frac{2n}{n^2-3}\right)^2 + 1\right]^{1/2}$, and so

$$x = 2\left[3\left(\frac{2n}{n^2-3}\right)^2 + 1\right]^{1/2}, \quad y = 3.2n/(n^2 - 3), \text{ from above.}$$

Ex.90 "If you have pride in gemotry, tell the triangles in which sides increase by unity, in many ways."

FORMATION OF INTEGRAL TRIANGLES UNDER GIVEN CONDITIONS

R.119-120 "3 being the length of the perpendicular and 4, the base of the first right angled triangle, and infinite (pairs of) right-angled triangles are produced in which sides increase by unity. (In these), the perpendicular from the vertex to the respective base is the sum of 'thrice the previous base added to the still previous perpendicular' and the base is twice the sum of 'the previous perpendicular added to the previous base.' Triangles in oppsition (in such triangles) are right angled and in all such triangles, 1 added to and substracted from the base, are the flank sides."

The rule gives a method of finding integral triangles in which each side and the base differ by unity and each such triangle is divided by the perpendicular from the vertex into a pair of integral right angled triangles. Let $b_1, b_2, b_3, \ldots$, be the bases, $a_1, a_2, a_3, \ldots$ and $c_1, c_2, c_3, \ldots$, the flank sides, and $p_1, p_2, p_3 \ldots$ the perpendiculars from the vertices of such triangles, in order. The rule states that if $a_1 = 3$ and $b_2 = 4$, and infinite number of triangles satisfying the above conditions can be formed by taking

$$p_n = 3b_{n-1} = p_{n-2} \ldots\ldots\ldots\ldots\ldots(1)$$

and $$b_n = 2(p_{n-1} + b_{n-1}) \ldots\ldots\ldots\ldots(2).$$

As sides differ by unity, we also have,

$$a_n = b_n + 1 \ldots..(3)$$

$$c_n = b_n - 1 \ldots\ldots(4)$$

Rational : With the notations and figure of the previous rule (R.118), as in that rule, we get, $\frac{3}{4}x^2 - 3 = y^2 \ldots (5)$ where x is the base and y, the perpendicular. $x = 2, y = 0$ is a solution of this equation. Also, the equation, $\frac{3}{4}x^2 + 3 = y^2$ has a solution, $x = 2, y$ 2.

Now, let the successive solutions of (5) be and $y = p_1, p_2, \ldots\ldots p_n$, in order. $x = b_1, b_2, \ldots\ldots, b_n$, Since, $x = b_1, y = p_1$ is a solution of (5) the next solutions of (5) will be,

$x\ (= b_2) = 2(b_1 + p_1)$ and

$y\ (= p_2) = 2p_1 + \frac{3}{4}\ b_1.2$ by the priciple of composition (**samāsa bhāvanā**)[1.5].

Similarly, again we get from this,

$x\ (= b_3) = 2b_2 + 2p_2$ and

$y\ (= p_3) = 2p_2 + \frac{3}{4} b_2 .\ 2$ and so on.

Thus, we have $b_{n-1} = 2(b_{n-2} + p_{n-2}) \ldots\ldots\ldots (6)$

and $p_{n-1} = 2(p_{n-2} + \frac{3}{2} b_{n-2}) \ldots\ldots\ldots (7)$

and so, $b_n = 2\ (b_{n-1} + p_{n-1}) \ldots.\ (8)$

Now, $p_n = 2\ (p_{n-1} + \frac{3}{2} .\ b_{n-1})$

$= 2[2(p_{n-2} + \frac{3}{2} b_{n-2})] + \frac{3}{2}\ b_{n-1}$, from (7)

$= p_{n-2} + 3(p_{n-2} + b_{n-2}) + \frac{3}{2} b_{n-1}$

$= p_{n-2} + \frac{3}{2} b_{n-1} + \frac{3}{2} b_{n-1}$ from (6)

$= p_{n-2} + 3b_{n-1} \ldots\ldots\ldots (9)$

Relations (8) and (9) establish the rule.

MISCELLANEOUS RULES

R.121 "For the mental satisfaction of good mathematicians and for the destruction of pride of bad mathematicians, rules told before and also not told earlier, are being told in this chapter on miscellaneous rules."

FORMATION OF RATIONAL TRIANGLES HAVING THE SAME HYPOTENUSE.

R.122-123 "The vertical line is called the upright, the horizontal line proceeding from the foot (of the upright) is called the base and the line touching the ends (of the two) is called the hypotenuse, by the mathematicians. The **hṛdaya** (radius) multiplied by 2 gives the diameter (of the circum-circle). The hypotenuse of a right angled triangle is equal to that. They (i.e., right angled triangles having the same hypotenuse) are produced from that. Equibilateral etc. quadrilaterals should be obtained from them."

R.80 (or R.81) gives a method for the formation of right angled triangles. In such a case, with the usual notations $h = 2R$, $a = 2.2Rn/(n^2 + 1)$ and $b = 2R(n^2 - 1)/(n^2 + 1)$, where R is the circumradius of the triangle. Several right angled triangles having the same hypotenuse is formed from that by changing the optional number. Cyclic quadrilaterals are formed from these triangles.

Ex.91 "O mathematician, tell me quickly, the equibilateral and the equitrilateral quadrilaterals, the measure of whose circum-radius is observed to be 325."

FORMATION OF EQUIBILATERAL TRAPEZIUMS HAVING THE SAME CIRCUMRADIUS.

R.124-125 "Greater side amongst two right angled triangles having the same hypotenuse is the diagonal (and) the smaller side is (each of) the flank sides. Divide the sum and the difference of the products of cross-multiplication of the sides and the uprights by the hypotenuse, separately. (The quotients) happen to be the base (and) the face of the equibilateral quadrilateral (in order). If the diagonal be less than the side, the diagonal and the flank side should be interchanged."

A pair of right angled triangles having the same circumdiameter is first formed by R.122-123 with optional numbers m and n.

Let $4Rm/(m^2 + 1) > 4Rn/(n^2 + 1)$

According to the rule a rational equibilateral trapezium can be formed such that each of its flank sides $= 4Rn/(n^2 + 1)$

$$\text{base} = 4R\left[\frac{m(n^2 - 1) + n(m^2 - 1)}{(m^2 + 1).(n^2 + 1)}\right]$$

$$\text{face} = 4R\left[\frac{m(n^2 - 1) - n(m^2 - 1)}{(m^2 + 1).(n^2 + 1)}\right]$$ and each of its diagonals $= 4Rn/(m^2 + 1)$

If the diagonal becomes less than the flank side, it shoud be interchanged with the flank side.

Rational : Obviously the quadrilateral will be a trapezium one. Its altitude will be $8R.m.n./(m^2 + 1)(n^2 + 1)$ by R. 36-39.

So, the circumdiameter of the trapezium $\left(= \frac{\text{flank side x diagonal}}{\text{altitude}}\right) = 2R.$

Therefore, all the trapeziums formed in this way will have the same circumdimeter. From one given circumdiameter, and infinity of right-angled triangles can be formed having the same hypotenuse by taking different values of

the optional number and so from them on infinity of equibilateral trapeziums can be formed having the same circum-diameter.

ALTERNATIVE RULE

R. 126-127a "Greater side amoing two right-angled triangles having the same hypotenuse is the base, the smaller side is (each of) the flank sides and 'the sum of the products of cross-multiplication of the sides and the uprights', divided by the hypotenuse is the diagonal. In the quadrilateral, the face should be obtained from the diagonals and the sides in the way described before."

With the notations of the previous rule (R124-125), the rule states that a rational equibilateral trapezium can be formed with the help of triangles having the same hypotenuse, such that, each of its flank sides = $4Rn/(n^2 + 1)$

base = $4\,Rm/(m^2+ 1)$

each of its diagonal = $\dfrac{4R\left[m\left(n^2-1\right)+n\left(m^2+1\right)\right]}{\left(m^2+1\right)\left(n^2+1\right)}$,

face = $\dfrac{4\,\mathrm{R}}{\mathrm{m}\left(\mathrm{m}^2+1\right)\left(\mathrm{n}^2+1\right)^2} = [\{m(n^2-1)+n(m^2-1)\}^2 - n^2(m^2+1)^2]$,

by R 100. Also. Altitude = $\dfrac{8Rn\left[n(m^2-1)+m(n^2-1)\right]}{(m^2+1).(n^2+1)^2}$,

As in the previous case, the circumdiameter of the trapezium = $2R$ and so the statement made in the last paragraph of the previous rule, is also true in this case.

FORMATION OF EQUITRILATERAL TRAPEZIUMS

R.127b-128a "Side of a right triangle is (each of) the flank sides and the base. After that, multiply twice the upright by the side. (The product) divided by the hypotenuse is (each) of the diagonals. The face should be obtained in the way described earlier."

With the notations of the previous rule, first of all a right angled triangle is formed, whose side = $4Rn/(n^2 + 1)$ and upright = $2R(n^2 - 1)/(n^2 + 1)$. According to the rule, a rational equitrilateral trapezium can be formed such that (each of) its flank sides = $4Rn/(n^2 + 1)$, base = $4Rn/(n^2 + 1)$ (each of) the diagonals.

$= \dfrac{8Rn(n^2-1)}{(n^2+1)^2}$. Also its face $= \dfrac{4Rn(3n^4-10n^2+3)}{3(n^2+1)}$ by R. 100, stated earlier

and its altitude $= \dfrac{16Rn^2(n^2-1)}{(n^2+1)^3}$ by R. 36-39.

The author of **GK** further comments, "If the face is greater than the base, the base and the face should be interchanged." As in the previous case, the circumdiameter of the trapezium = 2R and so the statement made in the last paragraph of R.124-125 is also true in this case.

RATIONAL SCALENE QUADRILATERALS FROM 3 RIGHT ANGLED TRIANGLES.

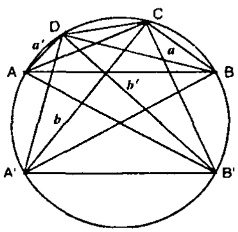


Fig. 13

R.128b-130a "Side of the first amongst (3 right-angled) triangles, having the same hypotenuse, is the base and those of the two others are the flank sides. Multiply the upright and the side of the first by the side and the uprights of the others (separately, in order). Divide the sum or at times, the difference also, of the products by the hypotenuse. (The quotients) are the (two) diagonals. Here also, the face of the scalene quadrilateral should be obtained in the way described before."

First of all, 3 right angled triangles, having the same hypotenuse and consequently the same circumdiameter are formed-by R. 122-123. Let the bases of these rectangles be, $4Rk/(k^2+1)$, $4Rm/(m^2+1)$ and $4Rn/(n^2+1)$ and their uprights be, $\frac{4R(k^2-1)}{(k^2+1)}$, $\frac{4R(m^2-1)}{(m^2+1)}$ and $\frac{4R(n^2-1)}{(n^2+1)}$, in order.

According to the rule, scalene quadrilateral ABCD can be formed, such that, its bases, $AB = 4R\,k/(k^2+1)$,

one of its flank sides, $AD = 4R\,m/(m^2+1)$,

the other flank side, $BC = 4R\,n/(n^2+1)$,

one of tis diagonals, $AC = \frac{4R\left[k(n^2-1)+n(k^2-1)\right]}{(k^2+1)(n^2+1)}$,

the other diagonal, $BD = \frac{4R\left[k(m^2-1)+m(k^2-2)\right]}{(k^2+1)(m^2+1)}$,

face, CD (obtained by R.100),

$$= \frac{4R}{k(k^2+1)(m^2+1)(n^2+1)} \cdot [\{k(n^2-1)+n(k^2-1)\}.\{k(m^2-1)+m(k^2-1)-mn(k^2+1)^2\}].$$

If the quadrilateral lies totally in semi-circle, the plus sign in the numerator of the diagonal, should be replaced by the minus sign. In such a case, another inscribed quadrilateral $A'B'CD$ can be obtained (as shown in the figure given above) having the same base, the same face and the same circumdiameter. The flank sides $A'D$ and $B'C$ of this quadrilateral are equal to the diagonals AC and BD of the previous quadrilateral, respectively. Let BC = a and $AD = a'$, such that, a, b and a', b' are the sides of the respective right angled triangles. Then, the diagonals $A'C$ and $B'D$ of this quadrilateral are given by b and b', respectively.

Rational : Let AB be the base and BC, a flank side of the scalene quadrilateral $ABCD$ (Fig. 13). Let another quadrilateral $A'B'CD$ be formed, whose base be on the other side of the centre and at the same distance from the centre, such that are CD = arc AA'. Then adding arc AD to both the sides, we get, arc AC = arc $A'D$. or chord AC = chord $A'D$. similarly, chord BD = chord $B'C$. Thus, $A'B'CD$ will be another quadrilateral in the same circle, whose base and face are the same as those of the quadrilateral $ABCD$ and whose flank sides are equal to the diagonals of the latter.

Now, let $A'B$ be joined, Then, due to symmetry, it must pass through the centre of the circumcircle (and so = h). Let $BC = a$, then from the right angled triangle $A'BC$, diagonal $A'C = b$. similarly, $B'D = b'$, since AD = a' and hence the rule.

FORMATION OF ISOSCELES AND SCALENE TRIANGLES HAVING THE SAME CIRCUMDIAMETER.

R. 130-b-c "Sides of two right angled triangles having the same hypotenuse are the two sides and the sum and the difference of the products of cross-multiplication of the upright and the side of one kept at two places with (with side and the upright of) the other' divided by the hypotenuse, are the bases."

Let two right angled triangles be formed. Let the base, the upright and the hypotenuse of the first be a,b and h and those of the second be a', b' and h, in order. Then, according to the rule, a scalene triangle can be formed, such that, its flank sides are a and a' and its base is $(a'b \pm ab')/h$. An isoscales triangle is formed if two congruent right-angled triangles are taken.

Ex. 92 "O friend, tell me quickly the issoceles and the scalene triangles, formed from two congruent or (two) non-congruent triangles (but) having the same hypotenuse (in order), if you have crossed the ocean of mathematics."

CYCLIC QUADRILATERAL AND THE THIRD DIAGONAL

Unless otherwise stated, the following notations have been used throughout this section in relation to a cyclic quadrilateral ABCD (for figure see R. 136).

a = base AB, b = flankd side BC

d = flank side DA, c = face CD

e = first diagonal AC, f = second diagonal BD

g = third diagonal, p = altitude DE, q = altitude CF

P= altitude of diagonal intersection point MN, R = circum-radius

D = Circum - diameter, A = area of the quadrilateal

i = lower segment AM of the first diagonal, j = upper segment MC of the first diagonal

k = lower segment BM of the second diagonal, l = upper segment MD of the second diagonal.

CIRCUM-RADIUS OF A CYCLIC QUADRILATERAL : ALTERNATIVE RULE

R.131 : "Product of the diagonals is multiplied by the product of the flank sides and divided by the product of both the perpendiculars. Half of the square-root of the quotient, happens to be the radius (of the circum circle, in case of) equibilateral, equitrilateral and inequilateral quardrilaterals."

That is $D^2 = bdef/pq$

Rational : In the quadrilateral ABCD (for figure see R. 136) We have, D $(= AD.BD/DE) = df/p$, (R.51)

Similarly, $D = be/q$. Therefore, $D^2 = bdef/pq$.

Ex. 93, and R.132 "The base of a quadrilateral is 630, face is 250, (flank) sides are 330 and 506, perpendiculars are 1584/5 and 2024/5 and diagonals are 624 and 520. O learned, tell quickly the circumradius. O mathematicians, if you are intelligent, have laboured hard and know to calculate with the rules of geometry, tell here quickly, the perpendicular, the area, the diagonals, the perpendicular from the point of intersection of the diagonals, horizontal and vertical segments of the diagonals, the circum-radius, the other perpendicular etc. and others also."

TO FIND CIRCUM-RADIUS: ALTERNATIVE RULE

R.133 "Subtract the product of the segments (of the base by the perpendicular) from the square of the perpendicular. Divide the difference by the perpendicular. Add the square of the quotient to the square of the base. Half of the square-root of the sum is the circum-radius."

That is, $R = (1/2).\left[a^2 + \left(\frac{p^2 - rs}{p}\right)^2\right]^{1/2} = (1/2)\left[a^2 + \left(\frac{q^2 - r's'}{q}\right)^2\right]^{1/2}$ r, s being segments of the base by the perpendicular ($DE=$) and r', s' being those of the same by the perpendicular CF $(= q)$.

Rational : By 51, we have (Fig.7) $R = AD.BD/2DE$ .... (1)

Now, $AD^2.BD^2 = (DE^2 + AE^2)(DE^2 + BE^2)$

$= (p^2 + r^2)(p^2 + s^2)$, where $AE = r$ and $BE = s$.

$= p^4 + p^2 a^2 - 2rsp^2 + r^2s^2$, since $r + s$ $(= AE + BE) = a$

$= p^2a^2 + (p^2 - rs)^2$

This gives, $AD.BD = p\left[a^2 + \left(\frac{p^2 - rs}{p}\right)^2\right]^{1/2}$

Therefore , from (1), $R\ (= AD.BD/2DE) = \frac{1}{2}\ p\left[a^2 + \left(\frac{p^2 - rs}{p}\right)^2\right]$

The other expression for R will be obtained, if we proceed with q in place of p.

Ex. 94 "What is the circum-radius of the triangle (whose) one side is 520, the other. 600, the base, 560 and the perpendicular (from the vertex), 480."

AREA OF A QUADRILATERAL

R. 134a "Multiply the sum of the products of the sides (of a quadrilateral) lying on the same side of a diagonal by the diagonal. Divide (the product) by 4 times the circum-raidus. (The result) is the area of equibiliateal and other quadrilaterals."

That is, $A = (ab + cd)e/4R = (ad + bc)f/4R$

Rational : Let r be the lenght of the perpendicular from B (Fig 4) to the diagonal AC $(=e)$., With usual notations, by R. 51, we get, $D = ab/r$. So, $r = (ab)/D$.

Now, area of the triangle $ABC = (1/2)\, AC.r = abe / 2D$.

Similarly, area of the triangle, $ADC \left(= \frac{cde}{2D}\right)$

By adding the two results, we get, area of the quadrilateral $ABCD = (ab + cd)e/4R$. Similarly, for the other part.

AREA OF THE TRIANGLE

R. 134b " Product of the sides divided by 4 times the circumradius, is the area of a triagle."

That is, area of a triangle $= abc/4R$, a,b and c being the sides of the trinagle.

Rational : Let ABC be a trignale (Fig. 3) and let its circum-diameter be D. Also, let CD $(=p)$ be dropped perpendicular from the vertex C to the base AB (for figure see R.36-39). Now, area of the triangle ABC $(=½AB.CD)$

$$= (½)\, c.ab/D, \text{ as } D = \frac{BC \, . \, AC}{CD} . = \frac{ab}{P}$$

$$= abc/4R \text{ as } D = 2R$$

CYCLIC QUADRILATERAL HAVING THE SAME AREA, CIRCUM-RADII AND DIAGONALS AS THOSE OF ANOTHER CYCLIC QUADRILATERAL.

R. 135 "Squares of the face, the base and (each of) the flank sides are separately subtracted from the square of twice the circum-radius. Square-roots of the differences are the face, the base and (each of) the flank sides of another quadrilateral (in order). Both the quadrilaterals have the same diagonal, the same area and the same circum-radius."

Let $ABCD$ be a given quadrilateral and let another quadrilateral be formed such that its base $= (4R^2 - a^2)^{½}$, its face $= (4R^2 - c^2)^{½}$, one of its flank side $= (4R^2 - b^2)^{½}$ and other flank side $= (4R^2 - d^2)^{½}$. Then, according to the rule, both the quadrilaterals have the same area, the same circum-radius and the same diagonals.

Rational : Let O be the circumcentre of the quadrilateral $ABCD$. Let a line $B'D'$ be drawn parallel to BD and lying at an equal distance (at which BD lies) from o, but on the opposite side of BD (from O). let AB', CB', CD' and AD' be joined. AP so, let BD' and $B'D$ be joined. Due to symmetry, these must pass through the centre o of the circum-circle (for figure, see R.145-147). Now, consider the quadrilateral $AB'CD'$. Since A, B', C and D' lie on the same circle on which A,B,C and D lie, the circum-radius of this quadrilateral will be the same as that of the quadrilateral $ABCD$. Also diagonal AC is common to both the quadrilaterals and by construction, $BD = B'D'$, being at the same distance from O, so their diagonals are the same. Also, triangle ABC and $AB'C$ are of equal area as they stand on the same chord and lie in the same segment. Similarly triangles ADC and $AD'C$ are of equal area. So, by adding, we find that both the quadrilaterals have the same area.

Now, since angle in a semicircle is a right angle, so

$AB' = (B'D^2 - AD^2)^{1/2} = (D^2 - d^2)^{1/2}$. Similarly, $B'C = (D^2 - C^2)^{1/2}$

$CD' = (D^2 - b^2)^{1/2}$ and $D'A = (D^2 - a^2)^{1/2}$ and hence the rule.

THIRD DIAGONAL OF A CYCLIC QUADRILATERAL, ALTERNATIVE RULE.

R.136 "4 times the circum-radius multiplied by the area (and then) divided by (the product of) the diagonals determines the other diagonal[116] which is obtained by interchanging the face with a flank side."

That is, $g = 4RA/ef$.

Rational : $ABCD$ is a cyclic quadrilateral whose sides are, $AB(=a)$, $BC(=b)$, $CD(=c)$ and $DA(=d)$ and whose diagonals AC and BD intersect at M, $DE\,(=p)$, $MN\,(=P)$ and $CF(=q)$ have been drawn perpendiculars from the points D,M and C to the line AB, and AS and CT those, from the points A and C to the diagonal BD, respectively and BQ, that from the point B to the diagonal AC. Let $AM=i$, $MC=j$, $DM=l$ and $MB=k$.

Let $ABC'D$ be another quadrilateral, such that, $BC' = CD = c$, Then, naturally $C'D=BC=b$ and its diagonal $AC' = g$(i.e. the third diagonal of the quadrilateral $ABCD$;$R.48$). Now, area of the quadrilateral $ABCD(=A$, say$) = (e/2R).\,(ab + cd)$ ...(1) by $R.134a$:

$AC = e$ $\quad BD = f$ $\quad AC' = q$

$DE = p$ $\quad CF = q$ $\quad MN = P$

$AM = i$ $\quad MC = j$ $\quad BM = k$ $\quad DM = l$

AS is perpendicular to BD

CT is perpendicular to BD

BQ is perpendicular to AC

Also from quadrilateral $ABC'D$, we get by R. 100

$gf = ab + cd$,...... (2)

From (1) and (2), we get the result. The concept of the third diagonal has enabled Nārāyaṇa to state several formulae in a very elegant form[117]. Earlier also, Nārāyaṇa has defined the third diagonal of a cyclic quadrilateral and given another

rule 118 to find its value. There he has also described the relation[119] between the three diagonals in different types of cyclic quadrilaterals and given another rule related to the third diagonal.

R. 137. R.137 is a repetition of R. 131.

CIRCUM-RADIUS; ALTERNATIVE RULE

R.138a "Alternatively, the product of 3 diagonals, divided by 4 times the area. is the circum-radius."

That is, $R = efg/4A$ (*cf. R.* 136)

Here, Nārāyaṇa quotes the rules given by Brahmagupta, Lalla and Śrīpati, in this respect and writes that their rules are wrong. According to them, half of the square-root of the sum of squares of opposite flank sides is the circum-radius of the inequilateral (quadrilateral.)

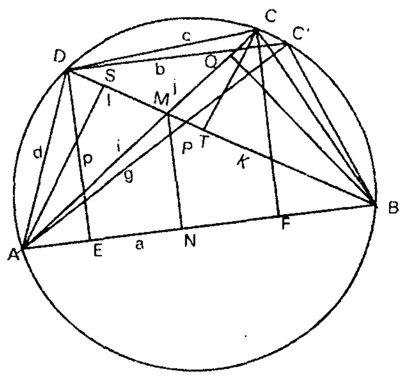


Fig. 14

SEGMENTS OF A DIAGONAL MADE BY THE OTHER DIAGONAL

R.138b-c "Take the products of the sides standing on a diagonl (on either side). (Severally) divide (these products) by their sum (and) multiply (the quotients) by the other diagonal. (The results) are the lower (and) the upper segments of (the other) diagonal (made by the former diagonal) from the point of intersection of the diagonals."

That is, $i = \frac{ade}{(ad+bc)}$; $j = \frac{bce}{(ad+bc)}$; $k = \frac{abf}{(ab+cd)}$ and $l = \frac{cdf}{(ab+cd)}$

Rational : In Fig. 14, let AS and CT be perpendiculars from A and C to BD. Then from similar triangles ASM and CTM, we have, $\frac{AS}{CT} = \frac{i}{j}$ .....(1)

Also, since D is the circumdiameter of the triangles, BCD and BAD, so by R.51,

$$D = \frac{bc}{CT} = \frac{ad}{AS}$$

or $$\frac{AS}{CT} = \frac{ad}{bc}\left(= \frac{i}{j}\right) \text{ from (1)}$$

or $$\frac{i}{ad} = \frac{j}{bc} = \frac{i+j}{ad+bc}\left(= \frac{e}{ad+bc}\right)$$

or $$i = \frac{ade}{ad+bc} \text{ and } j = \frac{bce}{ad+bc} \text{ ..... (2)}$$

Similarly $$k = \frac{abf}{ab+cd} \text{ and } l = \frac{cdf}{ab+cd}$$

SEGMENTS OF DIAGONALS: ALTERNATIVE RULE.

R. 139 "Divide the base and the face severally by the third diagonal (and multiply (the quotients) by the flank sides. (The results) are the lower (and) the upper segments of the (other) diagonals."

That is, $i = ad/g$, $j = bc/g$, $k = ab/g$ and $l = cd/g$.

Rational : We have $ab + cd = fg$ [Relation (2), R.136] and $ad + bc = eg$. Substituting the latter value in relation (2) of R.138 $b - c$, we get, $i\ (= ade/eg) = ad/g$, and $j = bc/g$. Similarly, $k = ab/g$ and $l = cd/g$.

PERPENDICULAR FROM THE TIP OF A DIAGONAL TO THE BASE

R.140 "Alternatively, multiply twice the area by the lower segment of a diagonal. (The product) divided by (the product of) the base and the diagonal gives (the length of) the perpendicular touching the tip of the other diagonal, to the base, in case of equibilateral and other quadrilaterals."

That is, $p = 2Ai/ae$ and $q = 2Ak/af$.

Rational : We have, $D = df/p$ (R. 51b) or $p = df/D$

So, $p = d.2A/eg$ [as $A = efg/2D$, (R.52a)] $= 2Ai/ea$ as $i = ad/g$, (R.139)

Similarly, $q = 2Ak/af$.

R.141. "Divide the third diagonal by the product of the circum-radius and twice the base. Also, severally multiply the two diagonals by the lower segments of each other. Multiply the products by the quotient. (The results) are the altitudes."

That is, $p = fi \times (g/2aR)$ and $q = ek \times (g/2aR)$.

Rational : The results follow from the relations $p = 2Ai/ea$ and $q = 2Ak/af$ (R.140), when $efg/4R$ is substituted for A (R. 52).

PERPENDICULAR FROM INTERSECTION OF THE DIAGONALS TO THE BASE

R.142. "Multiply the product of the lower segments of the diagonals by the product of the flank sides. Divide the square-root of the resulting product by twice the circum-radius. The result is (the length of) the perpendicular dropped from the point of intersection of the diagonals (to the base)."

That is, $P = (bdik/4R^2)^{1/2}$

Rational: From similar triangles AMN and ACF. (Fig. 14). We have, $P = qi/e = (ekg/2aR).(i/e)$, by R.141.

So, $P^2 = ik.ik.g^2/4a^2R^2$

$$= \frac{ad}{g} \cdot \frac{ab}{g} \cdot \frac{ik.g^2}{4a^2R^2} \text{ (R.139)}$$

$= bdik/4R^2$.

SEGMENTS OF THE BASE MADE BY SAID PERPENDICULAR.

R.142-143a "Severally subtract the square of the flank sides from the square of the circum-diameter and divide the square-roots thereof by the corresponding flank sides. These (i.e., the quotients) are called **Sakalas** (i.e., the components). The base divided by the sum of these components gives (the lenght of) the perpendicular from the point of intersection of the diagonals (to the base). This multiplied by the components gives the segments of the base by the perpendicular dropped from the point of intersection of the diagonals."

That is, $P = \dfrac{a}{\dfrac{\sqrt{D^2-b^2}}{b}+\dfrac{\sqrt{D^2-d^2}}{d}}$

$AN = \dfrac{P(D^2-b^2)^{\frac{1}{2}}}{b}$ and $NB = \dfrac{P.(D^2-d^2)^{\frac{1}{2}}}{d}$

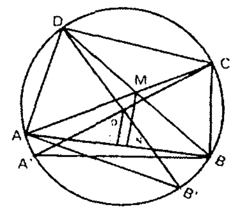


Fig.15

Rational :In Fig. 15 let DB' and CA' be the diameters of the circum-circle of the quadrilateral ABCD and let MN be perpendicular from the point of intersection of the diagonals of ABCD to its base AB. From similar triangles MNA and $A'BC$ (as $A'BC$ is a right angle). We get.

AN $\left(=\dfrac{MN.A'B}{BC}\right)=\dfrac{P.(D^2-b^2)^{\frac{1}{2}}}{b}$. Similarly, $NB=\dfrac{P.(D^2-d^2)^{\frac{1}{2}}}{d}$ adding the two results, we have,

$a=\dfrac{P(D^2-b^2)^{\frac{1}{2}}}{b}+\dfrac{P(D^2-d^2)^{\frac{1}{2}}}{d}$, from which we get desired P.

SUCĪ PERPENDICULAR, SUCĪ SIDES AND SEGMENTS OF BASE BY SUCĪ PERPENDICULAR.

R.143b-144a "Divide the base by half of the difference of squares of the base and the face. (The quotient) multiplied by the area (of the quadrilateral) happens to be the **sūcī** perpendicular. That (i.e., the **sūcī** perpendicular) multiplied by **sandhi** (i.e., a link) or a side and divided (severally) by their respective perpendiculars, happens to be the respective segments of the base (by the **sūcī** perpendicular) or the respective **sūcī** sides."

Let the sides AD and BC of the cyclic quadrilateral, $ABCD$ (Fig.7) be produced to meet at M and let MQ be drawn perpendicular to the base AB. The rule states that,

(1) **Sūcī** perpendicular (i.e., $MQ=p'$, say) $= Aa/\left(\dfrac{a^2-c^2}{2}\right)$

(2) Segments of the base by **sūcī** - perpendicular, i.e., $AQ = AE\ (p'/p)$ and $BQ=BF\ (p'/p)$

(3) **sūcī** sides, i.e., $AM\,(=s,\text{ say}) = \frac{p'd}{p}$ and $BM\,(=r,\text{ say}) = \frac{p'b}{q}$

Rational : From similar triangles ABM and CDM, $\frac{r-b}{c} = \frac{s}{a}$ or $r = \frac{(ab+cs)}{a}$ ....(1),

and $$\frac{(r-b)}{(s-d)} = \frac{s}{r} \quad ....(2)$$

Removal of r from (1) and (2) gives

$$s = \frac{a(ad+bc)}{(a^2-c^2)} \quad ....(3)$$

Now, from similar triangles AMQ and ADE, $p' = \frac{sp}{d}$

$= \frac{a(ad+bc)}{(a^2-c^2)}.\frac{P}{d}$, from (3)

$= \frac{a.4AR.P}{(a^2-c^2).f.d}$, as $A = \frac{f}{4R}(ad+bc).[R.134a]$

$= \frac{4a.AR}{(a^2-c^2).2R}$, as $\frac{fd}{p} = 2R[R.51]$

$= (Aa)/\left(\frac{a^2-c^2}{2}\right)$

The other results are easily derivable from similar triangles AMQ, ADE and BMQ, BCF. The author has given this rule in between rules on the third diagonal indicating that the rule is true only for cyclic quadrilaterals.

PERPENDICULAR FROM THE CENTRE OF CIRCUM-CIRCLE TO A SIDE OF THE QUADRILATERAL

R.144b. "The square-root of the difference of squares of half of a side (and) the circum-radius happens to be (the length of) the perpendicular from the centre (of the circum-circle to that side)."

In Fig. 15, Let OL be dropped perpendicular from the centre O of the circum-circle of the quadrilateral $ABCD$ to a side AB of it.

According to the rule, $OL = [R^2 - (a/2)^2]^{1/2}$

Rational : The relation is true as L becomes the midpoint of AB.

SEGMENTS OF DIAGONALS BY THE OTHER DIAGONAL

R.145-147 "The square-root of the difference of squares of a diagonal and the circum-diameter is called **Avakāśa.** The square-root of the difference of squares of the third diagonal and the circum-diameter is called **Guṇa**. Divide the sum and the difference of the cross-products of 'two **Avakāśas.** with the diameter and the **guṇa**' by the third diagonal (separately). (The quotients) are the difference of the segments of the diagonals (by the other diagonal). The

segments of the diagonals of both the quadrilaterals (by the other diagonal of the same quadrilateral) are obtained by the method of concurrence from the smaller diagonal with the smaller (sum or difference) and the bigger diagonal with the bigger (sum or difference)."

Rational : Let the diagonals AC and $B'D'(=f)$ of another quadrilateral $AB'CD'$ (Fig. 16) formed from quadrilateral $ABCD$ by R. 135, intersect at M' and let $AM'=i'$ " $M'C=j'$; $B'M'=k'$ and $M'D'=l'$. Also, let $u=(D^2-e^2)^{1/2}$, $v=(D^2-f^2)^{1/2}$ and $w=(D^2-g^2)^{1/2}$. Let AC and BD intersect as M. We also have, $Am=i$, $MC=j$; $BM=k$ and $MD=l$.

According to the rule,

$$i+j=e \text{ and } i-j=\frac{Dv+uw}{g}$$

$$k+l=f \text{ and } k-l=\frac{Du+vw}{g}$$

$$i'+j'=e \text{ and } i'-j'=\frac{Dv-uw}{g}$$

$$k'+l'=f \text{ and } k'-l'=\frac{Du-vw}{g}$$

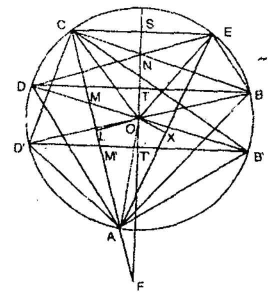


Fig. 16

Now i, j, k, l, i', j', k' and l' can be separately obtained from these relations by the method of concurrence (R.31, Chapter I)

Let O be the circum-centre of a cyclic quadrilateral $ABCD$. Let $BE=CD$ and let AE be joined. Thus AE is the third diagonal of the quadrilateral $ABCD$. Let the lines BC and DE meet at N, join ON and produce it to meet BD at T and CE at S and extend it to meet $B'D'$ at T' and CA produced at F. Due to symmetry and as $BE=DC$, the line ON a diameter of the circum-circle $O,B'ECD'$ will meet the lines, BD, CE and $B'D'$ perpendicularly at T, S and T', in order. Let OL and OX be perpendiculars from the centre O to the diagonal AC and the third diagonal AE, respectively.

$AM=i$	$AM'=i'$	OL is perpendicular to AC	$AB=a$
$MC=j$	$M'C=j'$	OX is perpendicular to AE	$BC=b=ED$
$BM=k$	$B'M'=k'$		$CD=c=BE$
$MD=l$	$M'D'=l'$		$DA=d$

From similar right-angled triangles OLF and OXE,

as $\angle LOF=180^0-\angle LOS$

$$=180^0-\frac{1}{2}(\angle COE+\angle COA)$$

$$=\frac{1}{2}\angle EOA=\angle EOX,$$

we have $$OF=\frac{OE.OL}{OX} \quad .....(1)$$

and $$\frac{OL}{LF}=\frac{OX}{XE} \quad ......(2)$$

From definition **avakāśa,** $u = (D^2 - e^2)^{1/2}$. so, $\frac{u}{2} = (AO^2 - AL^2)^{1/2} = OL$ ....(3)

$$v = (D^2 - f^2)^{1/2}. \text{ So, } \frac{v}{2} \; [= (OB^2 - BT^2)^{1/2}] = OT \quad .... (4)$$

Again, since, $(D^2 - g^2)^{1/2} = w$, so, $\frac{w}{2} = (OE^2 - XE^2)^{1/2} = OX$ ....(5)

Now, $TF = OF + TO = \left(\frac{Ru}{w}\right) + \frac{v}{2}$ as $OF = \frac{Ru}{w}$ ....(6)

From (1), (3) and (5) and as $OE = R$.

So, $TF = \frac{2Ru + vw}{2w}$ .... (7)

Let AC and BD meet at M,

Consider the triangle MFT, from symmetry,

$$MT = \frac{BM - DM}{2} = \frac{k - l}{2} \quad ...... (8)$$

Also, from similar right angled triangles OLF and MTF

$$\frac{MT}{TF} = \frac{OL}{LF}. \text{ So, } MT = \frac{TF.OX}{XE}, \text{ from (2), and so, } k - l = \frac{Du + vw}{g} \quad ..... (9),$$

from (5), (7) and (8), as $XE\ (= AE/2) = g/2$ and $D = 2R$.

Also, $k + l = f$ .... (10)

Therefore k and l can be separately obtained from (9) and (10) by the method of concurrence (R. 31, chap. 1),

Exactly in a similar manner,

$$i - j \ \frac{Dv + uw}{g} \text{ and } i + j = e.$$

So, again, i and j can be separately obtained.

Again, $T'F\ (= OF - OT') = \frac{2Ru - vw}{2w}$ from (4) and (6) as $OT = OT'$.

Therefore, by a process similar to above, like equation (9), we obtain,

$$k' - l' = \frac{Du - vw}{g} \quad ..... (13)$$

Also, $k' + l' = f$ ..... (14)

since $B'D'\ (= BD) = f$.

Similarly, $i' - j' = \frac{Dv - uw}{g}$ .....(15)

and $i' + j' = e$. .....(16)

Now, k' and l' can be separately obtained from equations (13) and (14) and i' and j' from (15) and (16).

Area of a Triangle Formed by Segments of Diangonal of a Cyclic Quadrilateral

R.148. "Multiply the product of segments of the diagonals forming the

triangle by the third diagonal. Divide (the result) by 4 times the circum-radius. (The quotient) happens to be the area of the triangle."

A cyclic quadrilateral is divided into 4 triangles by its diagonals. Area of any one of these triangles = $ikg/4R$, where i, k are segments of diagonals by the other diagonal of the quadrilateral, forming the triangle and similar formulae hold good for the other 3 triangles.

Rational : Consider the trianle MAB in Fig. 14. Let BQ be perpendicular from B to AM (produced, if necessary). Now, area of the triangle ABM (= $\frac{1}{2}.AM.BQ$) = $(\frac{1}{2})i.ab/2R$, as $AM = i$ and $2R = ab/BQ$ by R.51.

Thus, area of triangle ABM = $ikg/4R$, for $k = ab/g$ by R.139.

R.14\ "For fear of being lengthy I have described here the rules of geometry in brief. The learned should excuse me, if I have not stated surprising results."

REFERENCES

1. P.Singh's translation of **Gaṇita Kaumudī** Chapters I, II, and III was published in **Gaṇita Bhāratī**, Vol. 20(1998), pp.25-82.

2. Sarasvati Amma, T.A., **Geometry in Ancient and Medieval India**, Motilal Banarasi Das, Delhi, 1979, 1.

3-4. Datta, B.B. and Singh, A.N., "Hindu Geometry," **IJHS** Vol.15 (1980), 121.

5-7. R.C. Gupta., 'Vedic Mathematics from the Śulba Sūtras", **Indian Jour. of Mathematics Education**, Vol 9(2) (1989) 2-3.

8. Sarasvati Amma (ref.2) 6.

9. See also,**GSS**, 181.

10. See also, **PG**, 87

11. Cf **BSS**, chap, XVIII, vs.31; **SŚ**, 87; BBi, 7.

12. Cf. **BSS** chap XVIII, vs 32-33; **SŚ**, 87; BBi,8.

13. Cf. **BSS**, chap XVIII, vs.34; **SŚ**, 89; BBi, 8.

14. Cf. **BSS**, chap. XVIII, vs. 35-36a; **SŚ**, 89; **BBi**, 8-9.

15. Cf. **BSS**, chap. XVIII, vs. 36b; **SŚ**, 91; **BBi**, 9.

16. Cf. **GSS**, 182; **MS**, chap. XV, vs, .66.

17-19. Gupta, R.C., "Primitive Area of Quadrilateral and Averaging", **GB**, Vol.19(1997), 55.

20. Cf. **GSS**, 185.

21. Afzal Ahmad, "On Vedic Square-Circle Conversion", **GB**, Vol.4 (1982), 72.

22-23. Gupta, R.C., "Some Ancient Values of Pi and Their use in India", **ME**, Vol. IX(1) (1975), SEC.B, 1-2.

24. See also, **GSS**, 186.

25. See also, **GSS**, 190.

26. See **Tri**, vs. 47.

27.(a) See also, Shukla, K.S., "Hindu Methamatics in the Seventh Cenury A.D." **Gaṇita**, Vol. 22(2), 65.

(b) Cf. **Tri**, vs. 44.

28. Cf. **GSS**, 187.

29.(a) Cf. **GSS**, 189.

(b) See also, Gupta, R.C., "Areas of Regular Polygons in Ancient and Medieval Times," **GB**, Vol.16 (1994) 64-65.

30. See also, **GSS**, 189-190.

31. See also, **GSS**, 234-235.

32. Cf. **Tri**, vs. 51; **MS**, chap. XV, vs.54; **L**, 76.

33. Gupta, R.C., "Mathematics of the Mahāvedī," **Vishveshvaranand Indological Journal.**, Vol.XXII (1984),4.

34-35. **Ā**, Introduction, XLV-XLVI.

36. Gupta (ref. 5), 6.

37. Gupta (ref. 5), 5.

38. **BBi**, 13-14.

39. Pages 13-15, The work was ed. by K.S. Shukla, A.B. Sanskrit Parisad Lucknow, 1970.

40. Cf. **Tri**, vs. 46; **MS**, chap. XV, vs. 55; **SS'**, 54; **L**, 77-78.

41. Ref.1,32.

42. For Rule of Concurrence, Ref. 1,34-36.

43. See also Ref. 27a, 69.

44. See also, **L**, 86-87; **BBi**, 32.

45. Ref. 27a, 70.

46. Cf. **BSS**, chap. XII, vs, 21b; **GSS**, 193; **Tri**, vs. 43a;
PG, 90; **MS**, chap. XV, vs. 69; **SS'**, 43; **L**, 95.

47. Cf, **GSS**, 1993; **Tri.** Vs, 42b; PG, 88; **SS'**, 47; **L**,98.

48. Gupta, R.C., "Popularity of Jain Value of π in China and Japan" (in Hindi), **Arhat Vacana,** Vol. 4(1) (1992), 1-5.

49-50. Gupta R.C., "Jain Value of Π and its propagation in foreign countries (in Hindi); **Arhat Vacana,** Vol. 1(1) (1988), 15-18.

51. Ref. 48.

52.(a) Gupta, R.C., "Āryabhaṭa I's Value of Π", **ME,** Vol.7(1)(1973), Sec. B., 17-20.

(b) Gupta, R.C.,"Indian Values of the Sinus Totus", **IJHS**, Vol.13(1978), 125-143.

53. Ref. 52a and 52b.

54-55. CF. **GSS,** 193; **MS**, chap XV, vss, 76-77.

56. Cf. **MS,** chap. XV, vs. 78; **L,** 92-93.

57. Cf. **MS,** chap. XV, vs.64; **L,** 91-92.

58. Cf. **PG**,86.

59. Cf. **L**, 92.

60. Cf. **MS,** chap.XV, vs.70, **L,** 96-97.

61. Cf. **L,** 101-102.

62. Cf. **Tri**, vs.42a.

63. Cf. **L**, 103.

64. Cf. **BSS**, chap. XII, vs. 28; **GSS**, 194; **SS'**, 50.

65. Cf. **GSS**, 245; **SS'**, 48.

66. **L**, 106-107.

67. Gupta R.C., "Brahmagupta's Formulas for the Area and Diagonals of a Cyclic Quadrilateral", **ME**, Vol.VIII (2) (1974) Sec. B.36.

68. Ref. Ex.41. of this article.

69. Ref. R.36 and R. 38a of this article.

70. Ref. Exs. 45, 46 and 47. of this article.

71. Cf. **L**, 111-112.

72. Ref. Ex. 47. of this article.

73. See also, **GSS**, 237 and **L**, 113.

74. Cf., **L**, 113-115.

75. Ref., Ex. 47, of this article.

76. Cf., **L**, 90. See also, **GSS**, 236-237.

77-78. Ref. Ex. 47. of this article.

79. Cf. **MS**, chap. XV, vss. 98b-99; **SS'**, 55-57; **L**, 119-120.

80. Ref. 3, p.160; ref. 2, p.173-175.

81. **Ā**, 60-61; **BSS**, chap. XII, vs. 42; **GSS**, 250; **SS'**, 57.

82-83. Cf., **BSS**, chap. XIV, vs. 25-24; **L**, 123-124.

84. In his **Mahābhāskarīya** ed. and translated by K.S. Shukla, Lucknow Univ., 1960, 207-210.

85. See **IJHS**, Vol.18(1983), 71.

86. Gupta, R.C., "**Bhāskara** I's Approximation to Sine," **IJHS**, Vol.2 (1967), 121-136.

87. See also, Gupta, R.C., "The Līlāvatī Rule for Computing Sides of Regular Polygons," **ME**, Vol. IX(2) (1975), Sec. B. 25-29.

88. Cf., **PG**, 66-69.

89. **PG**, 67.

90. **PG**, 69-72.

91. Ref. Exs. 45-47. of this article.

92. Ref. 5,5.

93. Cf. **L**, 78-79.

94-95. Cf. **L**, 81.

96. Cf. **BSS**, chap. XII, vs. 33; **GSS**, 204; **L**, 82.

97. Cf. **GSS**, 204-205.

98. Ref. 134. for the method of concurrence.

99. See also, **BSS**, chap. XII, vs. 36.

100. See also, **BSS**, chap. VII, vs. 37 and. **GSS**, 208.

101. Cf. **BSs**, chap. XII, vs. 38; **GSS**, 209; **SS'**, 61-64 and **L**, 107-110.

102. Cf. **BSS**, chap. XII, vs. 34 and **GSS**, 212.

103. HHM, Part II, 227.

104. Ref., Ex. 77. of this article.

105. See also, GSS, 225.

106. Cf. GSS, 226.

107. Ref., R.34b of this article.

108. Cf. GSS, 218.

109. Ref., R.82 of this article.

110. Cf. GSS, 219.

111. Cf. GSS, 220.

112. HHM Part II, 225.

113. Cf. GSS 220.

114. Cf. GSS, 213.

115. GK, Part II, 232-233; See also, HHM, Part II, 174.

116. For other details refer, R.47-52 of this article,

117. Singh, P., Quadrilateral and Its Third diagonal, **Vishveshvaranand Indological Journal**, Vol. XXI, Part-i-ii, 1983, 219-227.

118-119. Ref., R.47-52 of this article.

> "One striking transformation typically experienced by students encountering the history of science for the first time is that they realize that the development of science has involved not only scientific creations, but also scientific creators, not only settled conclusions, but also intense controversies. This transformation helps studentss to see science as an intensely human enterprise."
>
> —Michael J. Crowe

4) *Līlāvatī Vyākhyā of Parameśvara* (Transcript by N. Ananta Krishna Sarma, No 295), Oriental Manuscripts Library, University of Kerala. (p 3-4).
5) *Kriyākramakarī* (p 9-10)
6) *Līlāvatī of Bhāskarāċārya with Vāsana of Pt. Damodara Miśra*, Edited by D. Jha, Mithila Institute, Darbhanga, 1959. (p 6-7).
7) *Bhāskarāċārya racita Līlāvatī* with Malayalam Commentary of P.K. Koru; Mathrubhoomi Printing and Publishing Co. Ltd. Kozhikode 1954 (p 10-11).
8) *Līlāvatī of Bhāskarāċārya* with Hindi Commentary by L. Jha. Edited by Pt. Sri Suresh Sarma, Chowkamba Vidhyabhavan, Varanasi 1976 (p 10-11)
9) *MahāSiddhānta of Āryabhaṭa II*, Edited with notes by Sudhakara Dvivedi, Benares, 1910. (p 143).
10) *Arithmetic, its Structure and Concepts* by Francis. J. Mueller, Brentice Hall Inc, New Jersey, 1960. (P 49 & 86).
11) *Arithmetic, A Modern Approach* by Bevan K. Youse, Prentice Hall Inc. New Jersey 1963. (p 19).
12) *History of Hindu Mathematics* Part I; B.B. Datta and A.N. Singh, Asia Publishing House, Bombay 1935 (p 133, foot note 2)
13) *ibid*. (p 132 foot note 1).
14) *Kriyākamakarī* (p.75)
15) *Līlāvatī* ed. by D.V. Apte (p. 35-36)
16) *Kriyākramakarī* (p. 75-77)
17) Koru (ref. 7 above) (p. 37)
18) *Dāmodara* (ref. 6 above) (p. 33)
19) *Atharva Veda* (V 28.6, V 1.9)
20) *Baudhāyana Śulba Sūtra* of Baudhāyana, Edited by G. Thibaut, Calcutta 1913. (I. 29-32, 42, 68, III 16-23, IV 6, 45, VI 6-10 etc.)
21) *Āryabhaṭiya* I, Edited by K.S. Shukla and K.V. Sarma, INSA, New Delhi 1976, (Gaṇita 27; p 70)
22) *Brāhmasphuṭa Siddhānta* of Brahmagupta Edited by Ramaswarup Sarma, Indian Institute of Astronomical and Sanskrit Research, New Delhi 1966. (XII. 2 (i)).
23) *Pāṭīgaṇita* of Sridhara, Translation by K.S. Shukla, Luknow University; 1959, (Sūtra 36, p 17)
24) *Gaṇitasāra Saṁgraha* of Mahāvira Edited with English Translation and Notes by M. Rangacharya, Govt. Press, Madras, 1912 (iii. 56)
25) *Gaṇita Tilaka* of Sripati, Edited by H.R. Kapadia; Gaikwad Sanskrit series No: 78, Baroda, 1935, (p 30, line 16)
26) *Siddhānta Śekhara* of Sripati, Edited by Babuaji Misra, Calcutta University, 1947 (XIII-11 (i))
27) *Kriyākramakarī* (p. 91).
28) Pāṭīgaṇita (ref. 23 above) Sūtra 21); *ibid* 25, (i. 49); *ibid* 9, (XV. 10. ii - 11.i) etc.
29) *Līlāvati* ed. by D.V. Apte (p . 39).
30) *Līlāvatī*, Koru (ref. 7) (p. 44).
31) *ṚgVeda* I. 164.11, 32.14, X 93.15, 97.1 etc.
32) *ṚgVeda* II 18.4, 18.5; *Atharva Veda* X 1-11; *Vājasaneyī Saṁhitā* 18. 24-25; *Taittrīya Saṁhitā* (VII 2. 11-19 etc.

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THE GAṆITA KAUMUDĪ OF NĀRĀYAṆA PAṆḌITA

CHAPTERS V-XII

ENGLISH TRANSLATION WITH NOTES

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TRANSLATOR'S INTRODUCTION [1*]

Chapters V to VIII of the *Gaṇita Kaumudī* deal with geometry. In it, first of all, rules to find the rough volume of an irregular excavation and the exact volume of a regular excavation, that of an excavation whose top is similar to the base and that of an acute solid have been given (R.1-4). After that *Pāṣāṇa hasta* has been defined (R.5.) and then a rule to obtain the volume and the surface area of a sphere has been dealt with (R.6.). Depth of an excavation has been treated in R.7(a).

A rule to find the volume of a stack and the number of bricks in it, is available in R.7 (b-c). Area of the sawn pieces of timber is the next topic of treatment (R.8-9). Rules to find the volume of a mound of grain have been given in R.10-11.

Relations between the part of the day passed (or remaining), the lenght of the gnomon and that of its shadow have been dealt with in R.12-13. Rules to find either the height of the lamp post or that of the gnomon or the distance between them or the length of the shadow (of the gnomon) when any 3 out of these 4 constituents are known, have been given in R.14-15. Cases of two gnomons and two lamps (in the same vertical line) have been dealt with in R. 16 and R.17-18, respectively.

The following abbreviations are used.

- Ā = *Āryabhaṭīya* of Āryabhaṭa I, edited and translated by K.S. Shukla and K.V. Sharma, INSA, New Delhi, 1976.
- Bi = Bhāskarācārya's *Bījagaṇita* and its Translation by S.K. Abhyankara, Bhaskaracharya Pratishthana, Pune, 1980.
- Bija = *Bījagaṇitāvataṁśa* of Nārāyaṇa Paṇḍita, edited by K.S. Shukla, Lucknow 1970.
- BSS = *Brāhmasphuṭa-siddhānta* of Brahmagupta, edited with S.Dvivedi's commentary, Vāsanā, and Hindi translation by R.S. Sharma *et al*, 4 Vols., New Delhi, 1966.

*All references to chapters V to IX and indicated by superscripts 1 to 100 are given at the end of Chapter IX — editor (RCG)

Ex. = Example (*Udāharaṇa* or *uddeśaka*)

GB = *Gaṇita Bhāratī.*

GK = *Gaṇita Kaumudī* of Nārāyaṇa Paṇḍita, edited by Padmakara Dvivedi, Benares, Part I, 1935; Part II, 1942.

GSS = *Gaṇitasāra-saṅgraha* of Mahāvīra, ed. with Hinditranslation by L.C. Jain, Sholapur, 1963.

GT = *Gaṇita-tilaka* of Śrīpati translated by K.N. Sinha, *GB*, 4 (1982), 112-133.

HHM = *History of Hindu Mathematics* by B.B. Datta and A.N. Singh, Lahore, Part I, 1935; Part II, 1938.

IJHS = *Indian Journal of History of Science.*

L = *Līlāvatī* of Bhāskara II, translated by H.T. Colebrooke, Kitab Mahal, Allahabad, 1967.

LBh = *Laghu-bhāskarīya* of Bhāskara I, Edited and Translated into English by K.S. Shukla, Lucknow, 1963.

LBh (G) = *Govindakṛti* of Bhaṭṭa Govinda (9th century A.D.).

Twenty- two stanzas dealing with the theory of pulveriser from the astronomical work *Govindakṛti* by Bhaṭṭa Govinda as quoted by Śaṅkara Nārāyaṇa (869 A.D.) and translated into Englsih by K.S. Shukla, form Appendix I (Pages 103-114) of *LBh.*

MBh = *Mahābhāskarīya* of Bhāskara I, edited and translated by K.S. Shukla, Lucknow, 1960.

ME = *The Mathematics Education* (Siwan).

MS = *Mahāsiddhānta* of Āryabhaṭa II, ed. by S.Dvivedi, Benares, 1910.

PG = *Pāṭīgaṇita* of Śrīdhara, ed. with an ancient commentary and transl. by K.S.Shukla, Lucknow, 1959.

R = Rule (*sūtra*) especially from the *GK.*

SŚ = *Siddhānta - śekhara* of Śrīpati, ed. by Babuaji Misra, Part II, Calcutta, 1947.

Tri = *Triśatikā* of Śrīdhara, ed. by S. Dvivedi, Benares, 1899.

vs., vss. = verse, verses.

GAṆITA KAUMUDĪ

CHAPTER V

EXCAVATIONS AND CONTENTS OF SOLIDS (*Khāta Vyavahāra*)

An approximate formula for the volume of a frustrum of a pyramid on a rectangular base is found in the *Śulba Sūtras* of Baudhāyana and others[2]. If (a, b) be the length and breadth of the base of a solid, (a', b'), the corresponding sides of the face parallel to it and h be its height, then

its[3] volume, $$V = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right) \times h.$$

Brahmagupta gave[4] an accurate formula for the same in the form,

$$V = P + \frac{1}{3}(G - P), \quad \text{where} \quad P = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right).h, \text{ and}$$

$$G = \frac{1}{2}(ab + a'b').h.$$

Thus, the formula becomes[5],

$$V = (h/6).\,[(a + a')(b + b') + ab + a'b'] \quad .....(1)$$

Here, it may be noted that Brahmagupta has not specified[6] the shape of the face and base of the excavation contemplated by him. Thus, if the face and the base be circular and if their radii be R and r, respectively, then by the rule,

$$V = (\pi h/3).\,(R^2 + r^2 + Rr),$$

after simplification[7]. If, $a' = 0 = b'$, then Volume of a cone (or pyramid) = 1/3 (base). (height) and so on. The Chinese mathematical classic *Chiu-chang Suan-shu* composed originally by Chang T'sang (died 152 B.C.) contains[8] an equivalent form of (1). However, Brahmagupta's rule is[9] more general. Heron of Alexandria (between 150 B.C. and 250 A.D.) also gave a formula for the volume of a truncated wedge.[10] Still earlier, the volume of frustum of a pyramid with square base were known in different cultural areas[11].

ROUGH CONTENT OF AN IRREGULAR EXCAVATION AND EXACT CONTENT OF A REGULAR ONE

R.1-2(a) "If the breadth, length or depth of an excavation is uneven, the sum of the measures divided by the number of steps (at which measures are taken) brings about equality. Multiply the area of the base by the depth. The product is the measure of solid cubits contained in the excavaton.[12]"

According to the rule

Volume = area of the base x depth.

Also, if the excavation be uneven, one should work with the average length, breadth or depth, as the case may be. Obviously that will give only the rough content of the excavation.

Ex. 1 "The length of a pond is 18 cubits and the breadth is 6 cubits. The depth is 3, 5 and 7 (at 3 places). O friends, tell the content of the excavation."

Ex. 2 "An excavation having equal base and face has, length equal to 4 cubits and breadth equal to 3 cubits, the depth being equal to 4 cubits. What is the content of this as well as that (of another excavation) having (each dimension) equal (to 4 cubits)?"

Exact Content of an Excavation Whose Base is Similar to the Top

R.2 (b)-3(a) "Divide the sum of the areas of the base, the face and (that of the rectangular figure formed by) the sum (of their lengths and that of their breadths) by 6. Multiply (the quotient) by the depth. (The product) is the content (of the ditch whose) face is similar to the base[13]."

According to the rule, the volume of the content of an excavation will be given by (1).

Rationale : Let $ABCD$ be the top and $EFGH$, the base of an excavation, both the base and the top being rectangular and plane. Let perpendiculars be drawn from E, F, G, and H to the top and also, from the feet of these perpendiculars to AB, BC, CD and DA as in the figure given below. Also suppose that $AB = a$, $BC = b$, $EF = a'$ and $FG = b'$.

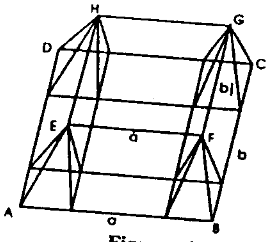


Figure - 1

Now, the excavation is divided into 3 parts (1) A rectangular parallelopiped at the centre whose volume is, $a' b' h$. (2) 4 equal pyramids on rectangular bases, one at each corner, their total volume being

$$(h/3).\ (a - a')\ (b - b')$$

(3) 4 triangular prisms on 4 sides, (a) 2 triangular prisms each of volume, $(h/2).(a - a')\ b'$, (b) 2 other triangular prisms each of volume, $(h/2).\ (b - b')a'$.

Therefore, $V = a'b'h + (h/3).\ (a - a')(b - b') + (h/2).\ (a - a')b' + (h/2)\ (b - b')\ a'$

$$= (h/6)\ [ab + a'b' + ab + a'b' + a'b + ab']$$

$$= (h/6)\ [(a + a')\ (b + b') + ab + a'b'].$$

The rule was earlier given by Brahmagupta, Mahāvīra, Śrīpati and Bhāskara II. The *Tiloyapaṇṇati* has[14] the rule for the volume of a prism with right triangular section. The volume of a cone or pyramid was[15] given by Brahmagupta, Nemicandra, Śrīpati and Bhāskara II.

Ex.3 "The dimensions of the bottoms (of 2 excavations) are 3,4 and 4,4, separately and those of the tops are seen to be those multiplied by 3 (each). The depth is 6.0 mathematician, if you know all the methods of measuring excavations, tell the contents of those 2 (excavations, separately)."

R.3(b)-4 "Divide the sum of the squares of 'the tip', 'the bottom' and 'their sum' by 8. The quotient multiplied by the depth is the rough volume of the well. One-third of the content of a solid with equal bottom and top happens to be the (rough) volume of the conical solid."[16]

According to the rule, if D and d be the diameters of the top and the bottom of a well of depth h,

its rough volume $= (h/8).\ [D^2 + d^2 + (D + d)^2]$

According to latter part of the rule, rough volume of the acute solid whose diameter at the top is D

$$= (h/24).\ [D^2 + D^2 + (D + D)^2] = (h/4)D^2.$$

Rationale : With the above notations, we have, area of the top $= (\pi/4)D^2$ and that of the bottom $= (\pi/4)d^2$ and the area taking diameter equal to the sum of the diameters $= (\pi/4)\ (D + d)^2$. Now by R.2(b)-3(a), volume of the solid $= (h/6)\ [(\pi/4)D^2 + (\pi/4)d^2 + (\pi/4)\ (D + d)^2] = (h/8).\ [D^2 + d^2 + (D + d)^2]$ by taking $\pi = 3$, for rough volume of the solid. For acute (i.e. conical) sold, take $d = 0$.

Ex.4 "The diameter of the top of a well is 16 cubits, that of the bottom is 4 cubits and the depth is 12 cubits. If you know, tell its content and the volume of such an acute solid."

While solving the problem, for finding exact values, Nārāyaṇa takes the value of π as 3.162 (as done earlier).[17]

R.5 "If the diameter or length of a solid is in *aṅgulas* (its) volume divided by 6144 happens to be (its volume) in *Pāṣāṇa-hastas.*[18]"

That is, 6144 cubic *aṅgulas* = *pāṣāṇa-hasta*

Ex. 5 "The length, breadth and thickness of a solid is $5\frac{1}{3}$, $3\frac{1}{2}$ and $\frac{1}{2}$ cubits (respectively). O elder, if (you) are an expert in *pāṭī (gaṇita)* and know, tell quickly, (its volume) in *pāṣāṇa-hastas.*"

Ex. 6 "The diameter of a cylinderical stone is 3 cubits (and its) thickness is half of that. O mathematician, tell quickly, (its volume) in *pāṣāṇa-hastas.*"

Volume and Surface Area of a Sphere

R.6 "Thrice the square of the diameter of a sphere is (its) exact surface area. Divide the surface area by 6. (The quotient) multiplied by the diameter, is the volume of a sphere.[19]"

According to the rule,

the surface area of a sphere of diameter $d = 3d^2$

and

its volume $= [(\text{surface-area})/6]d$.

Here, it may be noted that in the rule, Nārāyaṇa taking the value of π as 3, states the surface area of a sphere as exact. However, while solving the problem given below, he takes the value of π as 3 for calculating the rough surface area and the rough volume of the given sphere. For calculating the exact surface area and the exact volume, he takes the value of π as 3.162, as done earlier (*GK*, II, 199).

Problem 10 of the Moscow Mathematical Papyrus (ca.1850 B.C.) calculates the area of hemisphee like basket and some scholars believe that the then Egyptians knew[20] the correct formula $S = 2\pi r^2$, for the surface of a hemisphere.

Rationale : Divide the horizontal circumference of the sphere into parts equal to 4 times the number of R-sines in a quadrant. Now, divide the sphere into lunes by vertical circles the number of lunes being equal to the number of parts of the above horizontal circumference. Area of each lune should be found out by breaking it into n part again by horizontal lines at distances of 1 unit from each other.

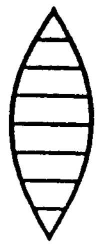


Figure - 2

Let l_n denote the length of the nth transverse arc. We have,

$l_n = \frac{T_n \times 1}{R}$, T_n being the nth R-sine.

So, area of a lune $= 2 \times \Sigma \frac{1}{2} (l_n + l_{n-1}) \times 1 = 2\,\Sigma(1/2R).(T_n + T_{n+1})$,

the summation being done so as to include all the R-sines. Therefore, area of a lune

$$= (2/R).(T_1 + T_2 + \ldots\ldots + T_{23} + \tfrac{1}{2}.T_{24}) = (2/R).(T_1 + T_2 + \ldots.. + T_{24} - R/2)$$

This is equated to the diameter of a sphere and verified with respect to the sphere of radius 3438 by actual calculation. So, area of one lune $= d$, d being the diameter of the sphere. Therefore, the whole surface[21] area $= d.\,\pi.d = \pi\, d^2$.

For volume of a sphere, pyramids equal to the units in the surface area, each of bases with unit side and altitude equal to the radius of the sphere by imagined on the surface of the sphere. Naturally the sum of the volumes of the pyramids will be equal to the volume of the sphere. But volume of one such pyramid $= \frac{1.d}{6}$.

So volume of all these pyramids $= \pi d^2.\ d/6 = \pi d3/6$ = Volume of the sphere.[22]

Āryabhaṭa II (950 A.D.?) was undoubtedly aware of a formula for the surface of a sphere.[23] Bhāskara II not only states the formula which is modern form amounts to the formula given above, but he has also explained the mode of derivation of the formula and an alternative process for the same.[24]

Approximate formula for the volume of a sphere has been given by Śrīdhara, Āryabhaṭa II and Mahāvīra.[25] Bhāskara II has not only given the correct formula but also given a rationale for the same.[25] In Europe it was Kepler who gve the above method of derivation of the expression for the volume of a sphere.[27]

Ex.7 "O friend, if you know, tell quickly the surface-area and the volume of a solid sphere of diameter 10 cubits, (in cubic cubits and in) *dr̥ṣat* (*Kara* as well)."

According to Nārāyaṇa, the rough surface area = 300 (square cubits) and the exact surface-area 316 $\frac{1}{5}$ (square cubits); the rough volume = 500 (cubic cubits) and the exact volume = 527 (cubic cubits). Also, the rough volume = 1125 *dr̥ṣatkara* and the exact volume = 1185 *dr̥ṣatkara* 4608 (cubic) *aṅgulas*.

DEPTH OF AN EXCAVATION

R.7(a) "The volume divided by the optional (or desired) area becomes the depth."

Ex.8 "Near the valley of a mountain, there is a square pond of (each) side (equal to) 5 cubits. A square stream (having its side equal to) 3 *aṅgulas* is falling from the mountain and fills the interior of the pond. O mathematician, tell the height of the pond."

CHAPTER VI-A

CITI VYAVAHĀRA (PILES OF BRICKS)

VOLUME OF A STACKAND THE NUMBER OF BRICKS

R.7(b-c) "The area (of the base of the stack) multiplied by the height becomes its volume.[28] (Thevolume of the stack) divided by the volume of a brick is the number of bricks in it."[29]

The stack is supposed to be in the form of a rectangular parallelopipedd. The rule is obvious.

Ex.9 "An altar of length, breadth (and) height 8, 6 (and) 3 (cubits) isconstructed of bricks of length, breadth (and) height 1, $\frac{1}{2}$ (and) $\frac{1}{4}$ cubit. Tell quickly, its volume (and) the number of bricks in it."

According to Nārāyaṇa," Alternatively, (the problem can) be solved by the rule of seven."

CHAPTER VI-B

KRAKACA VYAVAHĀRA

SAWN PIECES OF TIMBER

R.8 "Half of the sum (of the thicknesses) at the root and at the tip is multiplied by the length. (The product), multiplied by the number of section of the timber, is the measure in terms of *aṅgulas*. That divided by 576 happens to be the measure in (terms of) cubits."[30]

The object of the rule is to determine the sawyer's wages which is at a fixed rate per square cubit along which the sawing is done.

Rationale : The ends of the timber are supposed to be rectangular and its sections, trapeziums.

Let $ABCD$ represent one of the sections and EF, its length in the middle.

Then, its area = $\frac{1}{2}(BC+AD) \times$ length of the perpendicular betwen BC and AD,

$= \frac{1}{2}(BC+AD) \times EF$. The area multiplied by the number of sections gives the total area for which the payment is to be done.

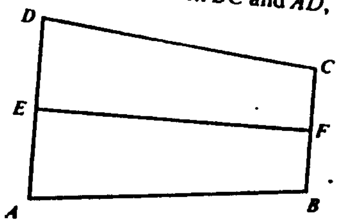


Figure - 4

Ex.10 "The measure (of thickness) at the root is 20 *aṅgulas*, (that) at the tip is 16 *aṅgulas*, the length of the log in the middle is 80 *aṅgulas* and (the number of) section is 6. If you are an expert in mathematics of timber, tell me quickly the result in cubits."

ALTERNATIVE RULE

R. 9 "If the wood is cut across, (the measure is found) by the multiplication of the thickness and the breadth, by the method stated earlier." (The sawyer's) wages (are settled) by agreement with the workman according to the sofiness (or) the hardness (of the mateial)."

Āryabhaṭa II has given the actal charges in cerain specific cases.[32]

Ex.11 "O wise mathematician, if you know, tell quickly the result in cubits of cross sawing of a log whose breadth is 27 *aṅgulas* and, thickness, 16 *aṅgulas*, the number of sections being, 10"

CHAPTER VII

RĀŚI VYAVAHĀRA (MOUND OF GRAIN)

VOLUME OF A MOUND

R.10 "Multiply the square of $\frac{1}{6}$ th of the circumference by the height. (The product) is the volume of the mound. Volume in (terms of) cubits, divided by 5, is (the volume) in[33] *Khārī*."

According to the rule, the volume of a mound = $(c/6)^2.h$, h being the the height of the mound and c, the circumference of its base.

A mound is supposed to be conical in shape. Therefore, its volume $= (\pi/3)(d/2)^2.h = (\pi^2 d^2 h)/12\pi = (c/6)^2.h$, taking $\pi = 3$.

According to latter part of the rule,[34] 1 cubic cubit = *khārī*.

Ex.12. "O learned, if you are an expert in calculations relating to mound, tell quickly in *Khārīs*, the measure of a mound whose circumference is 60 cubits and height, 6(cubits)."

Ex.13 "Friend, tell the measure of the mound of grain whose circumference is 21 cubits and height, 2 cubits and 8 *aṅgulas*."

VOLUME OF MOUND IN SPECIAL CASES

R.11 "Multiply the height by the square of one third of *vṛtti* (preimeter of the base i.e., the part of the circumference covered by the grain. (The product) divided by 2,1 and 3, is the volume (of a mound) piled against the side of a wall, (or) against the inside (or) against the outside corner (of it, respectively)."[35]

Along with above notations, if c_1, c_2 and c_3 be the perimeter of the base (i.e., the part of the circumference covered by the grain) when it is piled against the side of a wall or against the inside or the outside corner of it, then according to the rule, the volume of the mound $(C_1/3)^2.(h/2)$ or $(C_2/3)^2.h$ or $(C_3/3)^2.h/3$, in order.

Rationale : (a) When the grain is piled against the side of a wall, its volume $= (\frac{1}{2}).(C/6)^2 h$, by R.10 $= (C_1/3)^2.h/2$, as $C_1 = C/2$, in this case (see figure 4).

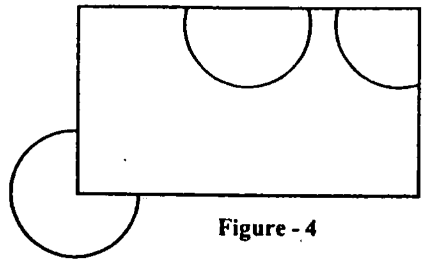


Figure - 4

(b) When the grain is piled against the inside corner of walls, its volume $= (\frac{1}{4}).(C/6)^2 h$, by R.10

$= (\frac{1}{4}).(4C_2/6)^2 h$, as $C_2 = C/4$, in this case (see figure 4)

$= (C_2/3)^2 h$.

(c) When the grain is piled against the outside corner of walls, its volume $= (3/4).(C/6)^2.h$, by R.10

$= (3/4).(2C_3/9)^2.h/3$, as $C_3 = (3/4).C$ in this case (see figure 4)

$= (C_3/3)^2.h/3$

Ex.14 "The circumference (i.e., *vṛtti*) of a mound resting against the inside corner is 15 cubits, (that of mound) resting against the side of a wall is 30 (cubits and that of a mound) resting against the outside corner is 45 (cubits). O learned, tell the measures of the mounds, quickly, if the height be 6 (cubits, in eachcase."

CHAPTER VIII

CHĀYĀ VYAVAHĀRA

SHADOW OF A GNOMON (MEASUREMENT OF HEIGHTS AND DISTANCES)

The *chāyā*, literally meaning shadow, is[36] a common topic for discussion in the Hindu treatises of mathematics. However, it also implies[37] the measure of time

and, of heights and distances therein. The topic formed[38] an important part of mathematics and astronomy from very early times. Even the *Sūryaprajñapati* (before 4th century B.C.) refers[39] to shadow lengths and their variations according to the time of the day and year. The mathematics section of the *Āryabhaṭīya* devotes[40] more than 3 verses to these problems.

PART OF THE DAY PASSED OR REMAINING

R.12 "Divide the (length of the) shadow by the height of the) gnomon. (The quotient) is called *Paurūṣī Prabhā* (i.e., the length of the shadow of the gnomon). Add 1 to that. Divide 1/2 by (the sum. The quotient) is the part of the day passed or the remaining part of the day (according as) the Sun is in the east or, in the west (respectively)."[41]

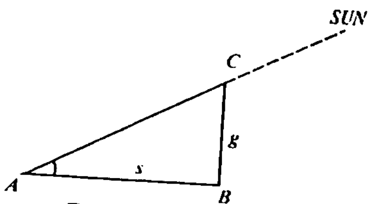


Figure - 5

According to the rule, if D/k be the kth part of the day elapsed (or remaining) since sunrise (or before sunset), $D/k = D/2\ (p+1)$, where

p (=*Paurūṣī*) = s/g, being the (length of the) shadow of the gnomon of height g.

Explanation : The rule is based[42] on the fact that in a right angled triangle for small values, angles are proportional to the opposite sides.

Let ACB be a right angled triangle, AC be the direction of the Sun, BC, the gnomon and AB, its shadow.

Then, $D/k = D/2\ (\cot A + 1) = D/2\ [s/g) + 1]$

The rule will give exact value for D/k when $A = 45^0$. For other values of A, it will give only approximate values.

Moreover, if the equinoctioal shadow is[43] not zero, the formula will be

$D/k = DG/2\ (s + g - e)$,

which takes the form of above formula for $e = 0$.

Ex.15 "The measure of the gnomon is 12 *aṅgulas* (and) the shadow is 4 times (that. O friend, tell me quickly the part of the day elapsed (and) with the earlier indications, (tell) the art of the day remaining."

R.13 "Divide half of the day by the part of the day elapsed (or) remaining. 1 subtracted (from the quotient) happens to be the *paurūṣī*. That multiplied by (the height of) the gnomon is (the length of) the shadow. (The length of) the shadow, divided by the *paurūṣī*, is (the height of) the gnomon."[44]

All the parts of the rule are easily derivable from R.12.

Ex.16 "Tenth part of the day has elapsed (or) remains to elapse. O mathematician, tell quickly (the length of) the shadow, if (the height of) the gnomon be 12 *aṅgulas* and from the part of the day (and the length of) the shadow, (tell the height of the gnomon)."

RELATION BETWEEN HEIGHT OF THE LAMP POST, THAT OF THE GNOMON AND ITS SHADOW.

R.14 "Divide the distance between (the foot of) the gnomon and that of the lamp (post) by (the difference of the height of) the gnomon subtracted from (the height of) the lamp. (The quotient) multiplied by (the height of) the gnomon is (the length of) the shadow.[45] "Multiply 'the distance, between (the foot of) the gnomon and that of the lamp (post)' by (the height of) the gnomon. Divide (the product) by (the length of) the shadow. (The quotient) added to (the height of) the gnomon happens to be (the height of)[46] the lamp."

With the above notations (R.12), according to the rule,

$$s = dg/(h - g) \text{ and } h = (dg/s)$$

d being the distance between (the foot of) the lamp (post) and that of the gnomon and h, the height of the lamp.

Rationale : Let AB (= h) be the lamp post, CD (= g) the gnomon and DE (= s), its shadow. From A draw AB (= h) perpendicular to meet ED produced at B and let BD = d Through D draw DF parallel to EA.

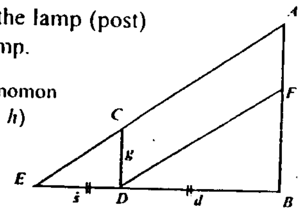


Figure - 5

Then, from similar triangles CDE and FBD, we have,

$DE/DC = BD/BF$ or $s/g = d/(h - g)$, as $CD = AF$.

From this we get the results.

Ex.17 "The distance between (the foot of) the lamp (post) and (that of) the gnomon is 2 cubits, the height of the lamp is $3\frac{1}{2}$ cubits and the height of the gnomon is 12 *aṅgulas*, Tell quickly (the length of) the shadow of that."

Ex. 18 "O foremost among mathematicians, if you know, tell the height of the lamp knowing the height of the gnomon and the distance 'between (the foot of) the lamp post and (that of) the gnomon' and (tell) the distance knowing (the heights of) the gnomon and the lamp and (also tell the height of) the gnomon knowing the distance and (the height of) the lamp."

DISTANCE BETWEEN THE FOOT OF THE GNOMON AND THAT OF THE LAMP POST AND THE HEIGHT OF THE GNOMON.

R.15 "Subtract (the height of) the gnomon from (the height of) the lamp. Multiply (the difference) by (the length of) the shadow. (The product) divided by (the height of) the gnomon is the distance between (the foot of) the gnomon (and that of) the lamp[47]

post." Multiply (the length of) the shadow by (the height of the lamp. (The product) divided by the sum of (the length of) the shadow and the distance between (the foot of) the lamp post (and that of) the gnomon is the height of the gnomon."

With the above notations, according to the rule,

$$d = s\,(h-g)/g \text{ and, } g = sh/(d+s)$$

which can be easily derived from the formula of R.14

The Base and the Height of the Lamp

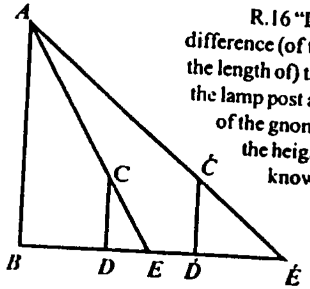


Figure - 7

R.16 "Divide the distance (between the tips) of the shadow by the difference (of the lengths) of the shadows. (The quotient) multiplied by the length of) the shadow is the base (i.e, the distance between the foot of the lamp post and the tip of the shadow). Multiply the base by the height of the gnomon. (The product) divided by the length of the shadow is the height of the lamp. The unknown should be obtained from the knowns using ratio (and proportion)".[48]

Let AB ($= h$) be the lamp post, CD ($= g$), $C'D'$ ($= g$), the gnomon and DE ($= s$), $D'E'$ ($= \acute{s}$), their shadows, respectively.

Also let $BD = a$, $BD' = \acute{a}$ and $ED' = b$.

According to the rule,

$$\text{(A)}\quad BE = \frac{E\acute{E} \times DE}{\acute{D}\acute{E} - DE}, \text{i.e.}, a + s = \frac{(b+\acute{s})s}{\acute{s}-s}$$

$$\text{(B)}\quad B\acute{E} = \frac{E\acute{E} \times \acute{D}\acute{E}}{\acute{D}\acute{E} - DE}, \text{i.e.}, \acute{a}+\acute{s} = \frac{(b+\acute{s})s}{\acute{s}-s}$$

$$\text{(C)}\quad AB = \frac{BE \times CD}{DE} = \frac{B\acute{E} \times \acute{C}\acute{D}}{\acute{D}\acute{E}}, \text{ i.e., } H = \frac{(a+s)g}{s} = \frac{(\acute{a}+\acute{s})g}{\acute{s}}$$

Rationale : From similar triangles ABE and CDE, we have, $AB/CD = BE/DE$ ... (1) and from similar triangles ABE' and $C'D'E'$, we get $AB/C'D' = BE'/D'E'$. . .(2)

From (1) and (2) theresult (c) immediately follows.

Also, from (1) and (2), $BE/DE = BE'/D'E'$. . . .(3)

as CD ($= g$) $= C'D'$

or $BE.D'E' = DE\,(BE + EE')$ from which the result (a) follows. Also, from (3), we have, $BE'.DE = D'E'(BE' - EE')$, from which the result (b) follows.

The rule was[48] earlier given by Āryabhaṭa I, Brahmagupta and Bhāskara II. Parameśvara (about 1360-1455), in his commentary on the *Āryabhaṭīya* has[49] given a rationale for the rule, based on the Rule of Three.

Ex.19 "The shadow of a gnomon of 12 *aṅgulas* is 5 *aṅgulas*. The shadow of the other gnomon kept in its front is 9 *aṅgulas*, the distance between the tips of the two shadows being 2/½ cubits. O my friend, if you are endowed with a number of good qualities and have the lamp of knowledge to dispel the darkness (of ignorance) and know, tell me the bases (i.e., the distances between the foot of the post and the tips of the shadows) and the height of the lamp."

Lengths of the Shadows and the Hypotenuses

R.17-18. "Divide 4 times the square of the gnomon by the difference of square of 'the difference of the shadows' and (that of) the hypotenuses'. Add 1 to (the quotient). Multiply thesquare-root of the sum by the difference of the hypotenuses (and, that of) the shadows (separately). (The products) happen to be thesum of the shadows (and that of) the hypotenuses, respectively. The shadows and the hypotenuses should be obtained from them by *saṁkramaṇa*. The heights of the lights should be obtained by the method stated earlier."

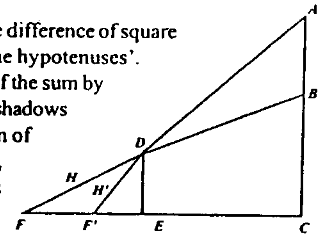


Figure -8

Let DE ($= g$) be the gnomon and suppose that the heights of the sources of light B and A are BC ($= h$) and AC ($= h'$) and the shadows of the gnomon are EF ($= s$) and EF' ($= s'$), respectively. Now, the hypotenuses are DF ($= H$) and DF' ($= H'$).

According to the rule,

$$s + \acute{s} = (H - H')\left[1 + \frac{4g^2}{(s-s')^2 - (H-H')^2}\right]^{1/2} \quad .....(1)$$

$$\text{and } H + H' = (s - s')\left[1 + \frac{4g^2}{(s-s')^2 - (H-H')^2}\right]^{1/2}$$

Now, separate values of s, s' can be obtained by the *saṁkramaṇa*[50] of (1) and given values of $s - s'$ and those of H, H'.

Also, now height of the source of light can be obtained by R.14 of this chapter.

Rationale : Let $s - s' = a$ (3) i.e., $s = s' + a$ (4) and $H - H' = b$ (5) i.e., $H = H' + b$ (6)

From the right angled triangles DEF and DEF', we have,

$$(H')^2 - (s')^2 = g^2 = g^2 = H^2 - s^2 \quad \text{as } H' = DF', F'E = s' \text{ etc.}$$

So, $(H')^2 - (s')^2 = (H' + b)^2 - (s' + a)^2$ from (4) and (6)

Solving this, we have $bH' = s'a + (a^2 - b^2)/2$

Squaring both the sides and then replacing

$(H')^2$ by $(s')^2 + g^2$, as $(H')^2 - (s')^2 = g^2$

and then simplifying, we get,

$$(s')^2 + as' + \left(\frac{a^2 - b^2}{-4}\right) - \left(\frac{b^2 g 2}{a^2 - b^2}\right) = 0$$

solving this quadratic equation in s', we get,

$$s' = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4.\left[\frac{(a^2 - b^2)}{4} - \frac{b^2 g^2}{(a^2 - b^2)}\right]}}{2}$$

Considering the positive value only, we find,

$$2s + a = \left[a^2 - a^2 + b^2\left(\frac{4b^2 g^2}{a^2 - b^2}\right)\right]^{\frac{1}{2}}$$

or $s' + s = b\left[1 + \frac{4g^2}{(a^2 - b^2)}\right]^{\frac{1}{2}}$ which is the same as (1) for $b = H - H'$ and $a = s - s'$

Similarly, from (7), we have, $bH' - \left(\frac{a^2 - b^2}{2}\right) = s'a$ Now, squaring both the sides and then replacing $(s')^2$ by $(H')^2 - g^2$, as $(H')^2 - (s')^2 = g^2$ and simplifying, we get the quadratic equation, $(H')^2 + bH' - \left(\frac{a^2 - b^2}{4}\right) - \frac{a^2 g^2}{(a^2 - b^2)} = 0$

Solving this quadratic equation for H' and then simplifying, we get (2) for $a = s - s'$, and $b = H - H'$

Ex. 20 "A (source of) light is fixed on the top of a pillar and the other is placed below it. The horizontal distance between the light and the gnomon is 70 *aṅgulas*. The measure of the gnomon is 12 *aṅgulas*. The measure of the distance between the tips of the shadows is 30 *aṅgulas* and the difference between (the lengths of) the hypotenuses is 24 (*aṅgulas*). O wise man, tell quickly the two hypotenuses, (the two shadows) and the heights (of both the sources of) light, if you are an experienced helsman to cross the ocean of mathematics."

Here, $s - s' (= a) = 30$, $H - H' (= b) = 24$ and $g = 12$

Now, $s + s' = 24\left[1 + \frac{4 \times 12 \times 12}{30 \times 30 - 24 \times 24}\right]^{1/2}$ from (1), $= 40$; also $s - s' = 30$,

so, shadows are, $s = 35$, and $s' = 5$.

Again $H + H' = 30\left[1 + \frac{4 \times 12 \times 12}{30 \times 30 - 24 \times 24}\right]^{1/2} = 50.$

Also, $H - H' = 24$, so hypotenuses, $H = 37$ and $H' = 13$.

Finally taking $d = 70$, from R. 14 of this chapter, we have, $h' = 180$ and $h = 36$

CHAPTER IX

INDETERMINATE EQUATION OF THE FIRST DEGREE

(i.e. $ax \pm c = by$)

INTRODUCTION (by the translator)

Ninth chapter of the *Gaṇita Kaumudī* deals with the indeterminate equation of the first degree (pulveriser), i.e., $ax \pm c =$ by. In it, first of all a rule to make the dividend and the divisor prime to each other, so that the equation may be solvable (in integers) has been given (R.19). Next the method of obtaining the minimum solution and then from it, the general solution of the pulveriser, in different cases, have been given (R.20-25). R.26 gives the solution when either the divisor and the additive or the dividend and the additive or both have a common (or different, in the last case) factor. A method of solution by working with the residue of division of either the additive or the dividend is negative has been dealth with in R.28(b)-29, whereas that when the additive is either an exact multiple of the divisor or cipher or the dividend is cipher, in R.30. In R.31, the solution of an equation with the help of that of an equation with unity as the additive or the subtractive has been given. After that 'the general problem on remainders' has been dealt with in R.32-33, the generalised conjunct pulveriser, in R.34 and an extension of the general problem on remainders, in R.35-36(a – b). R36(c) treats the topic of equal remainders. The solution of the problem when, in the rule of propertion, the argument, the fruit and only a meagre part of the fruit of the demand is given, with the help of a pulveriser, has been given in R.36(d)-39. Lastly, other 2 rules from Nārāyaṇa's *Bījagaṇitāvataṁsa* deal with the puveriser when its solution is fractional.

The topic of the solution of indeterminate equations of the first degree is generally called *Kuṭṭa*, *Kuṭṭaka*, *Kuṭṭākāra* and *Kuṭṭīkāra* etc. by the Hindus.

Problems which led the Hindus to the investigation of such equations broadly fall[51] into three varieties. In one variety the problem is to find a number (N) which when divided by two given numbers (a, b) will leave two given remainders (R_1, R_2). Thus, symbolically, $N = ax + R_1 = by + R_2$ or $by - ax\ (= R_1 \sim R_2) = \pm C$,

where $C = R_1 \sim R_2$.

In such type of problems (a, b) are[52] called 'divisors' (*bhāgahāra, bhājaka* and *cheda* (etc.) and (R_1, R_2), 'remainders' (*agra* and *śeṣa* etc.).

In the other variety, the problem is to find a number (x) such that its product with a given number (a) being increase or decreased by another given number (c) and then divided by a third number (b) will leave no remainder. Thus, symbolically, the problem is[53] to solve $(ax \pm c)/c = y$, in integers. In such type of problems, b is[54] generally called the 'divisor', a, the 'dividend' (*bhājya*), c, the 'interpolator' (*kṣepa* and *kṣepaka* etc.), the unknown quantities to be obtained, (x), the 'multiplier' (*guṇaka* and *guṇakāra* etc.) and y, the 'quotient' (*phala*). The third variety leads to the equations of the form, $by + ax = \pm c$.

Rule for the Solution of $ax \pm c = by$

R.19-20 "For the investigation of the pulveriser, if possible, divide the dividend, the divisor and the interpolater by a common divisor. If the number, by which the dividend and the divisor are divisible, does not divide the interpolator, the problem is incorrect.[55] Divide the dividend and the divisor, one by the other (and) the remainder of one by the remainder of the other. The (last) remainder is their common measure. Those two being divided by their common measure, become prime to each other".[56]

R.21-22 "Divide mutually the dividend and the divisor, made prime to each other, until the remainder is 1. Set down the quotients (of mutual division), one below the other, (successively), beneath them the interpolator and cipher below that. Multiply by the penultimate, the number just above it (and) add the ultimate. (to the product). Then, reject the ultimate. (Do so repeatedly) until only two number are left."[57]

R.23-25 "Divide the lower one by the reduced divisor and the upper one by like reduced dividend. The remainders are the multiplier (and) the quotient, in order. If the number (of mutual division be even (the remainders) so obtained are the multiplier and the quotient.[58] If that is odd, subtract the multiplier and the quotient, from their respective abraders. (If the interpolator is) positive, the remainders are the multiplier and the quotient but (if the interpolator is) negative, (these) subtracted from (their) respective abraders, (are so).[59] These, added to (the product of) any desired number multiplied by the abraders become manifold. In all methods of *Kuṭṭaka*, the learned should take the same (quotient) in (both the) abrading."[60]

Explanation : Consider the equation, $ax - by = \pm c$. Here, a is the dividend, b, the divisor and c, the interpolator (additive in case of $+c$ and subtractive in case of $-c$).

First if the dividend, the divisor and the interpolator have a common measure, these are divided by their common measure. If a number, divides the problem is incorrect. However, the condition is necessary only in case integral values of x and y are sought. In other cases, the condition is not necessary. (See 2 rules and example, given by Nārāyaṇa towards the end of this chapter).

Next, the (reduced) dividend is divided by the (reduced) divisor, after that the divisor by the remainder. Again the dividend is divided by the next remainder and so on. This process of mutual division is continued untill is obtained as a remainder. Thus a series of quotients are obtained. The interpolator is taken as the next number of this series and cipher, as the last number. These are kept one below the other, and thus a column of numbers (the quotients, the interpolator and cipher, in order) is obtained.

Now, the penultimate is multiplied by the number just above it and the ultimate is added to the product. The number penultimate is replaced by the sum and the ultimate is rejected. Thus another column of numbers is obtained. The process is repeated until only two numbers are left. The upper number is abraded by the reduced dividend and the lower number, by the reduced divisor. If the number of quotients of mutual division is even, the residue of reduction by the dividend is the quotient and the other residue, the multiplier. If the number of quotient of mutual division is odd, the upper number and the lower number (obtained by the process as in the above case), subtracted from the dividend and the divisor, are the quotient and the multiplier, respectively. If the interpolator is negative, the quotient and the multiplier are obtained by considering it as positive. The quotient and the multiplier, so obtained, are subtracted from the dividend and the divisor; the remainders are the quotient and the multiplier, respectively, for the negative interpolator.

In any one of the above 3 cases, any integral multiple of the dividend and the same integral multiple of the divisor, added to the quotient and the multiplier are also the quotient and the multiplier, respectively, for that case.

Rationale : Consider the equation, $(ax + c)/b = y$, i.e., $ax + c = by$ (A)
a being the dividend, c, the additive and b, the quotient. Also, suppose that a and b are[61] prime to each other and $b < a$. [If $b > a$ we will have, $q = o$ and $r_1 = a$; for q and r_1, see below.]

We perform the usual Euclideam algorithn on (or mutual division of) a/b.

Suppose we get

$a = bq + r_1,\ b = r_1 q_1 + r_2,\ r_1 = r_2 q_2 + r_3$

$r_2 = r_3 q_3 + r_4 \ldots\ldots\ r_{m-2} = r_{m-1} q_{m-1} + r_m,\ r_{m-1} = r_m q_m + r_{m+1}.$

Now, substituting the value of a in (A), we get,

$by = (bq + r_1).\,x + c$ i.e., $y = qx + y_1$, where $y_1 = (r_1x + c)/b$

Thus, as $a = bq + r_1$, so putting $y = qx + y_1$, (A) reduces to,

$by_1 = r_1x + c$ (1a)

Similarly, since $b = r_1q_1 + r_2$, so again putting $x = q_1y_1 + x_1$, (1a) reduces to

$r_1x_1 = r_2y_1 - c$ (1b) and so on.

Writing[62] down the successive values and the reduced equations, we have,

(1)	$y = qx + y_1$	(1a)	$by_1 = r_1x + c$
(2)	$x = q_1y_1 + x_1$	(2a)	$r_1x_1 = r_2y_1 - c$
(3)	$y_1 = q_2x_1 + y_2$	(3a)	$r_2y_2 = r_3x_1 + c$
(4)	$x_1 = q_3y_2 + x_2$	(4a)	$r_3x_2 = r_4y_2 - c$
(5)	$y_2 = q_4x_2 + y_3$	(5a)	$r_4y_3 = r_5x_2 + c$
(6)	$x_2 = q_5y_3 + x_3$	(6a)	$r_5x_3 = r_6y_3 - c$
....			
(2n–1)	$y_{n-1} = q_{2n-2}x_{n-1} + y_n$	(2n – 1)a	$r_{2n-2}y_n = r_{2n-1}x_{n-1} + c$
(2n)	$x_{n-1} = q_{2n-1}\,y_n + x_n$	(2n)a	$r_{2n-1}\,x_n = r_{2n}y_n - c$
(2n + 1)	$y_n = q_{2n}\,x_n + y_{n+1}$	(2n + 1)a	$r_{2n}\,y_{n+1} = r_{2n+1}\,x_n + c$

Now, suppose that the mutual division is [63] continued until the remainder is unity. (a) If the number of quotients be even, we have, $r_{2n} = 1$, $r_{2n+1} = 0$, $q_{2n} = r_{n-1}$, $y_n = q_{2n}x_n + c$ and $y_{n+1} = c$. If $x_n = t$, where t is any integral number, we get an integral value of y_n from which we can find out the value of x_{n-1} with the help of the equation (2n). Proceeding backwards step by step we ultimately find the values of x and y in positive integers.

(b) If the number of quotients be odd, we will have, $r_{2n-1} = 1$, $r_{2n} = 0$, and $q_{2n-1} = r_{2n-1} = r_{2n-2}$. The equations (2$n$ + 1) and (2n+1) a will be absent and the equations (2n) and (2n) a will be reduced to $x_{n-1} = q_{2n-1}y_n - c$ and $x_n = -c$, respectively.

Now, if $y_n = s$, s being any arbitrary integer, we will get an integral value of x_{n-1}. Then proceeding backwards, as above, we can ultimately find the values of x and y in positive integers.

(c) Let the interpolator be subtractive. Suppose that the equation is $ax - c = by$. Also, suppose that the solution of $ax + c = by$ is $x = m$, $y = n$, so that $am + c = bn$ i.e., $a(b - m) - c = b\,(a - n)$.

So, the solution of $ax - c = y$ will be $x = b - m$, $y = a - n$.

(d) General Solution. Let $x = m$, $y = n$ be a solution of $ax + c =$ by. (1) so that we have $am + c = bn$ i.e., $a(m + bt) + c = b\,(n + at)$ where t is any integer. Thus, if $x = m$, $y = n$ is a solution of (1), then $x = m + bt$, $y = n + at$ will also be a solution of (1).

Thus equation (A) is completely solved.

Āryabhaṭa I has[64] given a rule for solving $(ax + c)/b = y$. by stopping at any finite desired number of steps in mutual division of the dividend and the divisor. By considering the equation, $(ax - c)/b = y$, where y is a positive integer, Bhāskara I supplements[65] the form of Āryabhaṭa I, in which the interpolator is always made positive by necesary transposition. The same rule which holds under the same conditions has[66] also been given by Śripati. Bhāskara I has also given[67] a rule for solving $ax \pm c =$ by, where $a = mb \pm a'$ by proceeding at once with the solution of the equation $a'x \pm c = by$.

Brahmagupta has given[68] a rule for solving Āryabhaṭa's problem. Mahāvira has given[69] a rule with a view to find the solution of (ax $\pm$ c)/b = y, in positive integers. The same method has[70] been redescribed by him in a slightly different form. Here, in this case, he continues the mutual division until cipher is obtained as a remainder and further takes the optional multiplier to be cipher. Bhāskara II has also given a rule for solving $by = ax \pm c$.

Ex.21. "A number is multiplied by 273 and (the product is increased by 91. (The sum) is exactly divisible by 247. O friend, tell me quickly that number and also the quotient."

Solution : Here, the dividend = 273, the additive = 91 and the divisor = 247. These numbers have a common factor 13. Reducing them by 13, the dividend = 21, the additive = 7 and the divisor = 19, so the equation formed will be $21x + 7 = 19y$ (1)

The mutual division of the dividend and the divisor will be,

```
19)21 (1
   19
  ------
    2) 19 (9
       18
      ----
        1
```

so that the quotients obtained are 1 and 9. Also, the additive is 7. putting cipher at the end, the column formed will be,

1
9
7
0

Addition of the lowest, i.e., 0 to the product of the penultime i.e., 7 and the number just above it, i.e., 9, gives $(0 + 9 \times 7) = 63$. Now, the lowest 0 is rejected and the number 9 above the penultimate is replaced by 63. Thus, the new column will be

1
63
7

Again, by the same process, we get $(7 + 63 \times 1 =)70$, which replaces 1 and 7 is rejected. Thus the new column will be 70.
63

Since only two numbers are left, the upper one is abraded by the dividend, 21, and the lower one by the divisor, 19, and so the residues obtained are 7 and 6, respectively, which gives the minmum values of the quotient, y, and the multiplier, x, respectively, as the number of quotients of mutual division, (= 2) is even. Also, any integral number, multiplied by the divisor, 19, the product added to (or subtracted from) the multiplier 6, will also be a multiplier. Similarly, the same integral number multiplied by the dividend, 21, the product added to (or subtracted from) the multiplier 6, will also be a multiplier. Similarly, the same integral number multiplied by the dividend, 21, the product added to (or subtracted from) the quotient, 7, will also be the corresponding quotient. Thus, the minimum solution of (1) is $x = 6, y = 7$ and its general solution is[71]

$$x = 6 \pm 19t, y = 7 \pm 21t, \text{ where t is any integer.}$$

Here, it may be noted that the quotient of abrading the upper and the lower number of the column should be the same (R.25). If they are different, the residuec should be obtained after making the quotients equal.[72]

Alternative Rule

R.26 "The divisor and the additive' or 'the dividend and the additive' or 'both' may be reduced by a common (or different in the last case) divisor. The multiplier and the quotient (obtained), multiplied by their respective divisors, are the (true ones)"[73]

Explanation and Rationale : Let the equation be,

$$ax \pm c = by \qquad \ldots (1)$$

(a) Suppose that p is the common measure of a and c, such that

$$a = a'p \text{ and } c = c'p \qquad \ldots (2)$$

Let, $x = m, y = n$ be a solution of (2), we have,

$$a'm \pm c' = bn \text{ or } a'pm \pm c'p = bpn$$

or

$$am \pm c = bpn \text{ as } a'p = a \text{ and } c'p = c$$

and so $x = m, \ y = pn$ will be a solution of (1)

(b) Suppose q is the common measure of b and c, such that $b = b'q$ and $c = c'q$.

Let $x = m, y = n$ be a solution of $ax \pm c' = b'y$ (3)

Then, according to the rule, $x = qm, y = n$ will be a solution of (1).

Since, $x = m, y = n$ is a solution of (3) we get,

$$am \pm c' = b'n \text{ or } aqm \pm c'q = b'.qn \text{ or } aqm \pm c = bn$$

as $c = c'q$ and $b = b'q$ and so $x = qm, y = n$ will be a solution of (1).

(c) Lastly, suppose that p is that common measure of a and c such that $a = a'p$ and $c = c'p$, and q is the common measure of b and c (and so of b and c'), such that $b = b'q$ and $c' = c''q$.

Let $x = m, y = n$ be a solution of $a'x \pm c'' = b'y$ (4)

The, according to the rule $x = qm, y = pn$ will be a solution of (1)

Since, $x = m, y = n$ is a solution of (4), we have

$$a'm \pm c'' = b'n \text{ or } a'pqm \pm c''pq = b'qpn \text{ or } aqm \pm c'' bpn$$

as $a'p = a, c''pq = c'p = c$ and $b'q = b$ and so $x = qm, y = pn$ will be solution of (1).

Ex.22 "A number is multiplied by 80.30 is either added to or subtracted (from the product). (the sum) becomes exactly divisible by 39. Tell that number and) the quotient, separately."

Solution : Here, the dividend = 80 and the interpolator = 30 and so the equation will be

$$80x \pm 30 = 39y \qquad \ldots (1)$$

Dividing the divisor and the interpolator by their common measure 3 and again dividing the dividend and the reduced interpolator by their common measure 10, we have, the dividend = 8, the divisor = 13 and the interpolator = 1. and so, the equation will be

$$8x \pm 1 = 13y \qquad \ldots (2)$$

Mutual division of the dividend and the divisor will give the quotients 0, 1,1,1,1, in succession. The interpolator is 1 and so the column formed will be [0 1 1 1 1 1 0].

The final 2 numbers obtained by the successive reduction of number of numbers in the column by the method stated earlier will be [3 5]

Since the number of quotients of mutual division (= 5) is odd, subtracting 3 and 5 from the corresponding abraders 8 and 13, we get 5 and 8. Multiplying them by the common measure 10 of the dividend and the reduced interpolator and, 3 of the divisor and the interpolator, separately and in order, we get, the quotient ($= y$) = 50 and the multiplier ($= x$) =24. So the general solution of $80x + 30 = 39y$ will be

$$x = 24 + 39t \text{ and } y = 50 + 80t, \ t \text{ being any integer.}$$

In case of subtractive, since the number of quotients of mutual division is odd, dividing immediately, the final two numbers in the column (i.e.,3 and 5) by 8 and 13, respectively, we find that the residues are the same. Thus, multiplying as above, we get the least solution of $80x - 30 = 39y$ as $x = 15$ = 30 and so the general solution of this equation will be

$$x = 15 + 39t, \ y = 30 + 80t, \ t \text{ being any integer}$$

Alternatively, the problem can also be solved without one or the other or both reductions.

Ex. 23 "A number is multiplied by 5; 37 is added to or subtracted from the product. (the result) is exactly divisible by 3. Tell that (number) quickly."

Here the equation is $5x \pm 37 = 3y$.

The column formed will be [1 1 37 0]

Lastly, the two numbers left will be [74 37] Now, abrading of the first number 74, by the dividend 5, will give the quotient 14 and abrading of the last number 37, by the divisor 3, will give the quotient 12. Since, the quotients are different, we must take the residues with the same quotient. Thus, the multiplier ($= x$) =1 and the quotient ($= y$) = 14, (if the quotient of abrading is taken as 12) or $x = -5$, $y = 4$ (if the quotient of abrading is taken as 14). Moreover, here Nārāyaṇa comments,

"The practice of (taking) the same (quotient) for the abrader means any optional (ntegral) multiple of the abrader added to (the multiplier and the quotient), the multiplier and the quotient become manyfold and (also) any optional (integral) multiple of the abrader subtracted (from the two numbers), the multiplier and the quotient become manyfold."

Alternative Rule

R.27-28a "Abrade the interpolator by the divisor (and work with the residue in place of the interpolator). The quotient is the sum or the difference of the result (i.e., the quotient) of abrading (added to or subtracted from, the quotient obtained from working with the residue.)[74] Abrade the dividend by the divisor (and work with the residue in place of the dividend). The quotient is the sum (of the quotient obtained from working with the residue and the prduct of) the multiplier and the result (i.e., the quotient) of abrading."[75]

Explanation and Rationale : Let the equation be $ax \pm c = by$ ...(1)

Case. I. Suppose that $c = bq + s$ ...(2)

and let $x = m$, $y = n$ be the solution of $ax \pm s = by$ ...(3)

According to the rule, $x = m$, $y = n \pm q$ will be a solution of (1).

As $x = m, y = n$ is a solution of (3), so

$am \pm s = bn$ or $am \pm (bq + s) = b(n \pm q)$ Therefore $x = m$, $y = n \pm q$ will be a solution of (1) due to (2).

Case II. Suppose that $a = bp + r$ ....(4)

and let $x = m$, $y = n$ be the solution $rx \pm c = by$ ... (5)

According to the rule, $x = m$, $y = n \pm mp$ will be the solution of (1)

As $x = m$, $y = n$ is a solution of (5), so, $rm \pm c = bn$. or $(bp + r)m \pm c = b(n + mp)$.

Therefore, $x = m$, $y = n + mp$ will be a solution of (1) due to (4).

As has been done in the solution of Ex.24 by Nārāyaṇa, cases I and II can also be combined. Thus, suppose that $a = bp + r$ ....(6) and $c = bp + s$ ....(7) and also, $x = m$, $y = n$ be the solution of $rx \pm s = by$ ....(8).

Then, $x = m$, $y = n + mp \pm q$ will be the solution of (1).

As $x = m$, $y = n$ is the solution of (8), so $rm \pm s = bn$ or $(pb+r)m \pm (bq+s) = b(n+mp \pm q)$

Therefore, $x = m$, $y = n + mp \pm q$ will be the solution of (1) due to (6) and (7).

Ex. 24 "A number is multiplied by 1000. (when) 2210 is either added to or subtracted from the product. (the sum or the difference becomes exactly divisible by 143. Tell them, separately."

Solution : Here the equation is $1000x \pm 2210 = 143y$ (1)

Case I. Consider the equation, $1000x + 2210 = 143y$ (2)

Its solution will be $x = 65$, $y = 470$.

Alternatively, division of 1000 by 143 gives, the quotient = 6 and the remainder = 142. Solution of $142x + 2210 = 143y$

gives $x = 65$, $y = 80$. So the solution of (1) will be $x = 65$, $y (= 80 + 6 \times 65)$ = 470.

Alternatively, division of 2210 by 143 gives, the quotient 15 and remainder 65. Solution of $1000x + 65 = 143y$ gives $x = 65$, $y = 455$. So the solution of

(1) will be $x = 65, y = 470$.

Alternatively, division of 1000 and 2210 by 143 gives the quotients, 6 and 15 and the remainders, 142 and 65, respectively. Solution of

$142x + 65 = 143y$ gives $x = 65, y = 65$. So, the solution of (1) will be

$x = 65, y (= 65 + 6 \times 65 + 15) = 470$.

Case II. Consider $1000x - 2210 = 143y$ (2).

Its solution will be $x = 78, y = 530$. Alternatively, division of 1000 by 143 gives the quotient = 6 and the remainder = 142.

Solution of $142x - 2210 = 143y$ gives $x = 78, y = 62$.

So, the solution of (2) will be $x = 78, y (= 62 + 6 + 78) = 530$

Alternatively, division of 2210 by 143 gives the quotient = 15 and the remainder = 65. Solution of $1000x - 65 = 143y$ gives, $x = 78, y = 545$. So, the solution of (2) will be $x = 78$, and $y = (545 - 15) = 530$.

Alternatively, division of 1000 and 2210 by 143 gives the quotients, 6 and 15, and the remainders, 142 and 65, respectively. Solution of $142x - 65 = 143y$ gives, $x = 78, y = 77$. So, the solution of (2) will be, $x = 78$, $y (= 77 + 78 \times 6 - 15) = 530$.

NEGATIVE DIVIDEND

R.28(b)-29 "In the case of a negative dividend, supposing it as positive, find the multiplier and the quotient. One of these, either the smaller one or the greater one, should be made negative (and) at the same time, the other remains positive."[76]

Let $x = m, y = n$ be a solution of $ax \pm c = by$. Then, according to the rule $x = -m, y = n$ or $x = m, y = -n$ will be a solution of $-ax \pm c = by$.

The rule is obvious.

Ex.25-26 "(A number) is multiplied by -30; 3 is added to or subtracted from (the product). (The result) is exactly divisible by 7. Tell me (the quotient and) the multiplier. (A number) is multiplied by (-30); 97 is added to or subtracted from (the product). (The result) is exactly divisible by 7. Tell me quickly, (the quotient and) the multiplier."

SOLUTION WHEN ADDITIVE IS EXACT MULTIPLE OF DIVISOR.

R.30 "In case the additive is either an exact multiple of the divisor or cipher (or) the dividend is cipher, (then) the multiplier is cipher and the additive as divided by the divisor is the quotient."[77]

According to the rule, $x = 0$, y c/b is a solution

of $ax + c = by$, if, $a = 0$, or $c = 0$ or c/b = an integer.

The rule is obvious.

Ex.27-29 "(A number) is multiplied by 7; 9 is either added to or subtracted from (the product. The result) is exactly divisible by 3. Tell the quotient and the multiplier. A number is multiplied by 9. Cipher is added to (the product. The sum) is exactly divisible by 5. O mathematician, if you know, tell the quotient (and the multiplier). (A number) is multiplied by cipher. 12 is added to or subtracted from (the product). (The result) is exactly divisible by 4. O mathematician, tell me the multiplier (and the quotient)."

Here, Nārāyaṇa comments,

"In case, the dividend is cipher, the quotient is always constant (and) excluding cipher and infinity, it is possible for the multiplier also to be integral in all cases."

SOLUTION OF $ax \pm c = by$ FROM $ax \pm 1 = by$

R.31 "Find the multiplier and the quotient by supposing unity as the additive or the subtractive. These are multiplied (severally) by the desired additive or subtractive (made positive and then) abraded by their (respective) abraders. (The residues) happen to be (the multiplier and the quotient, in order)."[78]

Explanation and Rationale :Let $x = m, y = n$ be the solution of $ax \pm 1 = by$.

According to the rule, $x = cm, y = cn$ will then be the solution

of $ax \pm c = by$. ...(1)

The least solution of (1) will be obtained from this by abrading these values of x and y and their respective abraders.

As $x = m, y = n$ is a solution of $ax \pm 1 = by$, so,

$am \pm 1 = bn$, or $a(cm) \pm c = b(cn)$ and hence the rule.

GENERAL PROBLEM ON REMAINDERS

R.32-33 "Supposing the first divisor to be the divisor, the next divisor to be the dividend and the difference of the first remainder subtracted from the other remainder to be the additive, find the quotient. Multiply it by the first divisor and add (the product) to the first remainder. (After that), suppose (the sum to be the remainder (and) the product of the first divisor (and) the next divisor, the divisor, (Now), these become the first (remainder and the first divisor) and, also, that next to it, the other (and so on and thus finally, the remainder happens to be the (desired) number."

Explanation and Rationale : The rule gives a method to find N, which when separately divided by $a_1, a_2, a_3, \ldots\ldots$, leaves $r_1, r_2, r_3 \ldots\ldots$ as remainders in order, i.e., to find N such that,

$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = a_4x_4 + r_4$ ....(1)

For definiteness, let us consider, $N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3$ ...(2)

According to the rule, let us first solve

$(N =)\ a_1x_1 + r_1 = a_2x_2 + r_2$

i.e., $[a_2x_2 + (r_2 - r_1)]/a_1 = x_1$ ...(3)

Let $x_1 = m$, $x_2 = b_2$ be the minimum solution of this equation. Then, the least value of N satisfying (3) will be $a_1m + r_1$. Hence the general value of N satisfying (3) will be $N = a_1(a_2u + m) + r_1$, where u is any integer

$= a_1a_2u + a_1m + r_1$.

Now, consider, $a_1a_2u + (a_1m + r_1) = a_3x_3 + r_3$

i.e., $[a_3x_3 + (r_3 - a_1m - r_1)]/a_1a_2 = u$ ...(4)

Let $u = n$, $x_3 = b_3$, be the minimum solution of this equation. Then, the least value of N satisfying (4) will be $a_1a_2n + a_1m + r_1$. This value of N will also satisfy (3), as this value has been obtained from the values of N satisfying (3). Therefore, the general value of N satisfying (3) and (4) simultaneously, i.e., satisfying (2) will be given by,

$(N =)\ a_1a_2a_3v + a_1a_2n + a_1a_2n + a_1m + r_1$, where v is any integer ...(5)

Proceeding in this way, we will obtain the values of N satisfying (1).

The process of solution of equation (1) was[79] known to Āryabhaṭa I. Bhāskara I has interpreted[80] Āryabhaṭa I's rule as a method for the solution of this equation. Brahmagupta has[81] also given the rule. Bhāskara II has[82] given two methods for the solution of this equation. One of his method is the same as that given[83] by Āryabhaṭa I. One of Bhāskara II's problem was[84] later on dealt with by Ibn - al - Haitam (c. 1000) and Leonardo Fibonacci of Pisa (c. 1202).

Ex. 30 "O mathematician, tell me quicly (the numbers) which when dividend by 3,4,5 and 6 leave the remainders 2,3,4 (and) 5, (in order)."

Solution : The equation formed will be,

$N = 3x + 2 = 4y + 3 = 5z + 4 = 6t + 5$ ....(1)

Least value of x satisfying the equation, $3x + 2 = 4y + 3$, i.e. $(4y + 1)/3 = x$, will be 3. So, general value of x, satisfying this equation will be $4m + 3$, where m is any integer (R.25). Substituting this value of x in $3x + 2$, we get, $12m + 11$.

Now, consider the equation, $12m + 11 = 5z + 4$, i.e. $(5z - 7)/12 = m$. Least value of m satisfying this equation will be 4 and so its general value will be $5n + 4$, where n is any integer. Substituting this value of m in $12m + 11$, we get, $60n + 59$.

Finally, consider the equation, $60n + 59 = 6t + 5$, i.e. $t - 9 = 10n$

Least value of n satisfying this equation will be cipher and so, its general value will be p, where p is any integer. Substituting this value of n in $60n + 59$, we get, $60p + 59$ as general solution of (1), its least solution being59.

Examples like (1) are now known as 'the Chinese problems on remainders'. The first such problem occurs[85] in a work of Sun Tsu, who could get[86] only one solution of it. Even by the time of 6th and 7th centuries, three successive Chinese mathematicians obtained only 3 successive solutions. According to Mikami the Chinese interest in indeterminate analysis grew[87] after their contact with the Hindu culture and under the influence of the latter.

Ex. 31 "What are the numbers which when lessened by 4 and 3 or increased by 7, are exactly divisible by 7, 10 and 9, in order."

Generalised Conjunct Pulveriser (*Saṁśliṣṭa Kuṭṭaka*)

R.34 "Supposing the multiplier to be the dividend, the divisor to be the dividend, the divisor to be the divisor and the remainder to be the additive, obtain the multipliers by the method of *kuṭṭaka* (i.e., by the method of solution of indeterminate equation of the first degree). These (multipliers) are the corresponding remainders of their own divisors. (The desired number should be obtained from them) by the method stated earlier."

Explanation and Rationale : Let $b_1y_1 = a_1N \pm c_1$ ...(1)

$b_2y_2 = a_2N \pm c_2$. ...(2)

$b_3y_3 = a_3N \pm c_3$ ...(3)

Then, according to the rule, the value of N can be obtained from the equations,

$N = b_1y_1 + m = b_2y_2 + n = b_3y_3 + p = \ldots$ where $m, n, p, \ldots$ are the least values of N satisfying equations (1), (2), (3), ..., in order.

With the above natations, $b_1y_1 + m, b_2y_2 + n, b_3y_3 + p, \ldots$ will be the general values of N satisfying equations (1), (2), (3),...., in order (R.25) and hence the rule.

Mahāvīra,[88] Śrīpati[89] and Bhāskara II[90] have given rules for the solution of such type of equations. However, some mathematicians doubt the genuineness of the rule given by Bhāskara II while some others take a different view.[91]

Ex.32 "O mathematician, if you are just like a lion residing in mountains, to win the elephants, among the learned mathematicians knowing Indeterminate Analysis, tell quickly the numbers which when multiplied by 9, 7, 5 and 3 and divided by 5, 8, 11 and 14 leave remainders 1, 2, 3 and 4, in order."

Here the problem is to find the values of N which satisfy all the following equations, simultaneously.

$$9N - 1 = 5x \quad 7N - 2 = 8y \quad 5N - 3 = 11z \text{ and } 3N - 4 = 14t$$

Solving these equations, we find that the least values of N (i.e., the multipliers) are 4,6,5 and 6, respectively. Thus, according to the rule, we have to find the value of N, where N satisfies the equations,

$$N = 5x + 4 = 8y + 6 = 11x + 5 = 14t + 6.$$

By R.32-33, $N = 2414 + 3080m$, where m can take any integral value.

General Problem on Remainders (Continued)

R.35-36 (a-b) "Obtain the quantities (i.e., the divisor and the remainder) by the method stated earlier. The quotients are the first (divisor) and the first remainder. Desired divisor and the (desired) quotient are the second. Obtain the quotients from these by *kuṭṭaka* (i.e., by the method of Indeterminate Analysis). Multiply the quotient by the (initial) divisor. Add its own (i.e., initial) remainder to the product. The sum is the next remainder. Multiply the (initial divisor) by the other divisor. (The product) is the next divisor. In this way, other (divisor and remainders) should be obtained by repeating the process."

Explanation and Rationale : The rule is a generation of R.32-33 and gives several other solutions of the equation,

$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \ldots\ldots\ldots\ldots\ldots$ ....(1)

For definiteness, let us consider,

$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3$ ...(2)

We have seen in R.32-33 that relation (5) of that rule,

i.e., $N = a_1a_2a_3v + a_1a_2n + a_1m + r_1$ ...(3)

is a solution of (2) m, n and v having meanings as attached to them in that rule.

Now, divide $a_1a_2a_3\, v$ and $a_1a_2n + a_1m + r_1$ by a_1. This will give a_2a_3v and $a_1a_2n + a_1m + r_1$ by a_1. This will give a_2a_3v and $a_2n + m$ as quotients. Consider the equation, $a_2a_3v + a_2n + m = a_1x_1 + r_1$

Let $v = f$ and $x_1 = d_1$ be the minimum solution of this equation. According to the rule, $N = a_1^2\, a_2a_3w + a_1a_2a_3f + a_1a_2n + a_1m + r_1$

where w is any integer. ...(5)

This value of N will also satisfy (2) as this value has been derived from (3) which satisfies (2).

Again suppose that q_2 is the quotient when $a_1m + r_1$ is divided by a_2. Obviously, the remainder will be r_2 in this case as (3) satisfies (2) as well. So,

$a_1m + r_1 = a_2q_2 + r_2$ ...(6)

Now, division of $a_1^2\; a_2a_3w$ and $a_1a_2a_3f + a_1a_2n + a_1\, m + r_1$ by a_2 will give the quotients $a_1^2\; a_3w$ and $a_1a_3f + a_1n + q_2$.

Let $w = g, x_2 = d_2$ be the minimum solution of

$a_1^2 a_3w + a_1a_3f + a_1n + q_2 = a_2x_2 + r_2$ ....(7)

Then,

$N = a_1^2 a_2^2 a_3k + a_1^2 a_2a_3g + a_1a_2a_3f + a_1a_2\, n + a_1m + r_1$ ....(8)

where k is any integer. This value of N will also satisfy (2) as this value has also been derived from (3) which satisfies (2). Finally, suppose that q_3 is the quotient when, $a_1a_2n + a_1m + r_1$ is divided by a_3. Obviously, the remainder in this case will be r_3 as (3) satisfies (2) as well. So,

$a_1a_2n + a_1m + r_1 = a_3q_3 + r_3$ ...(9)

Now, division of $a_1^2a_2^2a_3k$ and $a_1^2a_2a_3g + a_1a_2a_3f + a_1a_2n + a_1m + r_1$ by a_3 will give the quotients $a_1^2a_2^2k$ and $a_1^2a_2g + a_1a_2\, f + q_3$.

Let $k = h, x_3 = d_3$ be the minimum solution of

$a_1^2a_2^2k + a_1^2a_2g + a_1a_2f + q_3 = a_3x_3 + r_3$ ...(10)

Then,

$N = a_1^2a_2^2a_3^2j + a_1^2a_2^2a_3h + a_1^2a_2a_3g + a_1a_2a_3f + a_1a_2n \pm a_1m + r_1$ ...(11)

where j is any integer. This value of N will also satisfy (2) as this value has been derived from (3) which satisfies (2) and so on and hence the rule.

Here, it may be noted that all the solutions [i.e., values of N given by (5) of R.32-33 or (5), (8) and (11) of R.35-36] of the equation (1) of R.32-33 or, R.35-36 are easily transferable from one form to the other by simply interchanging the optional numbers, suitably. Thus, if we take optional number w in place of v such that $v = a_1w + f$, where f is the number satisfying equation (4) of R.35-36, we get solution (5) of R.35-36 in place of solution (5) of R.32-33 of the equation (2) of R.32-33 (or, R.35-36) and so on.

Ex.33 "Tell the numbers which when separately divided by 3,5 and 7, leave the remainders 1,3 and 5, (in order)."

Solution : That is, find the value of N such that $N = 3x + 1 = 5y + 3 = 7z + 5$ . . .(1)

We have seen in R.32-33 that $N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3$

has the general solution. $N = a_1a_2a_3v + a_1a_2n + a_1m + r_1$, where v is any integer and $x_1 = m$, $x_2 = b_2$, is the minimum solution of $a_1x_1 + r_1$ $a_2x_2 + r_2$ and $u = n$, $x_3 = b_3$, is the minimum solution of $a_1a_2u + (a_1m + r_1) = a_3x_3 + r_3$.

So, in this case $N (= 3 \times 5 \times 7v + 3 \times 5 \times 6 + 3 \times 4 + 1) = 105v + 103$

will be general solution (1), for the minimum solution of $3x + 1 = 5y + 3$ is $x = 4$, $y = 2$, and the minimum solution of $3\times5u + (3\times4 + 1) = 7z + 5$ is $u = 6$, $z = 14$. . . .(2)

Now, we apply R.35-36. We divide $105v$ and 103, by 3, separately and get $35v$ and 34 as the quotients. Solving $35v + 34 = 3x + 1$, we get, $v = 0$, $x = 11$ as its minimum solution. So,

$N (= 3^2\times5\times7w + 3\times5\times7\times0 + 3\times5\times6 + 3\times4 + 1)$ where w is any integer, $= 315w + 103$

will also be the general solution of (1). . . . (3)

Again we divide $315w$ and 103 by 5, separately and get $65w$ and 20 as the quotients. Solving, $63w + 20 = 5y + 3$, we get, $w = 1$, $y = 16$ as its minimum solution. So,

$N(=3^2\times5^2\times7k + 3^2\times5\times7\times1 + 3\times5\times7\times0 + 3\times5\times6 + 3\times4 + 1)$ where k is any integer.

$= 1575k + 418$

will also be the general solution of (1). . . .(4)

Againg division of $1575k$ and 418 by 7 given $225k$ and 59 as quotients. Solving $225k + 59 = 7z + 5$ we get $k = 2$, $z = 72$ as its minimum solution. So,

$N = 3^2\times5^2\times7^2\times j + 3^2\times5^2\times7\times2 + 3^2\times5\times7\times1 + 3\times5\times7\times0 + 3\times5\times6 + 3\times4 + 1$

$= 11025j + 3568$ . . . . . (5)

will be the general solution of (1).

Ex.34. "Two quantities when multiplied by 3 and 5, and (then) divided by 5 and 7, leave 1 and 2 as the reminders, (in order). Their difference when multiplied by 4 and (then) divided by 9, leaves 5 as the remainder. Their sum when multiplied by 3 and (then) divided by 5, leaves 2 as the remainder. The sum of (the above quotients, when multiplied by 10 and (then) divided by 11, leaves 7 as the remainder. O friend, tell them."

That is, to find the value of x and y such that,

$3x = 5r + 1$ . . . . (1a) $5y = 7s + 2$ . . . (1b)

$4(y - x) = 9t + 5$ . . . . (1c) $3(x + y) = 5v + 2$ . . . (1d)

and $10(r + s + t + v) = 11w + 7$ . . . (1e), x, y, r, s, t, v and w being integral.

Solution : The solution of the problem as solved by Nārāyaṇa is on the following lines.

Solving (1a) by R.19 – 25, we have

$x = 2 + 5k$ . . . .(2a) and $r = 1 + 3k$ . . . . (2c)

where, k is any integer.

Similarly, solving (1b), we get

$y = 6 + 7p$ . . .(2b) and $s = 4 + 5p$ . . . . (2d)

where p is any integer.

Now, if we take $k = p$, relations (2a) and (2c) transform into

$x = 2 + 5p$ . . .(2g) and $r = 1 + 3p$ . . . (2h), respectively.

Also, then the values of x, y, r and s given by relations (2g), (2b), 2h) and (2d) respectively, will satisfy both the equations (1) and (1b).

Again with the help of (2g) and (2b), we obtain from (1c), $9t - 11 = 8p$

Solving it, we have,

$t = 3 + 8q$ . . . (3e) and $p = 2 + 9q$

where q is any integer. With this value of p, (2g), (2h), (2b) and (2d) transform into,

$x = 12 + 45q$ . . . (3a) $r = 7 + 27q$ . . . (3c)

$y = 20 + 63q$ . . . (3b) and $s = 14 + 45q$ . . . (3d), respectively.

Now the values of x, y, r, and s given by relations (3a), (3b), (3c) and (3d), respectively, will satisfy all the three equations, viz, (1a), (1b) and (1c).

Again, with the help of (3a) and (3b), we have from (1d), $5v - 94 = 324q$.

Solving this, we get, $v = 278 + 32m$ . . . (4f) and $q = 4 + 5m$

With this value of q, (3a), (3c), (3b) (3d) and (3e) transform into

$x = 192 + 225m$ . . . (4a) $r = 115 + 135m$ . . . . (4c) $y = 272 + 315m$ . . . (4b)

$s = 194 + 225m$ . . . (4d) and $t = 35 + 40m$ . . . (4e), respectively.

Also the values of x, y, r, s and t given by relation (4a), 4b), (4c), (4d) and (4e), respectively, will satisfy all the four equations, viz., (1a), (1b), (1c) and (1d).

Finally, with the help of (4c), (4d), (4e) and (4f), we get from (1e)
$11w - 6213 = 7240m$

Solving this equation, we have,

$w = 1225 + 7240m$, and $m = 1 + 11n$, where n is any integer.

With this value of m, (4a), (4b), (4c), (4d), (4e) and (4f) transform into,

$x = 417 + 2475n$ . . . (5a) $y = 587 + 3465n$ . . . (5b) $r = 250 + 1485n$ . . . (5c)

$s = 419 + 2475n$ ...(5d) $t = 75 + 440n$ ...(5e) and $v = 602 + 3564n$ ...(5f), respectively.

With the values of x, y, r, s, t and v given by relations (5a), (5b), (5c), (5d), (5e) and (5f), respectively, all the equations (1a), (1b), (1c), (1d) and (1f) are satisfied, simultaneously.

By taking $n = 0$, 1 and 2, in succession, we get,

$x = 417$ 2892 5367

$y = 587$ 4052 7517 and so on.

EQUAL REMAINDERS

R.36c "In the case of equal remainders, the remainder is the number and the least common multiple of the divisors, is the additive."

According to the rule, if, $N = ax + c = by + c = dz + c = \ldots$

then, $N = c$ with the L.C.M. of $a, b, d, \ldots$, as additive. The rule is obvious.

Ex.35 "O friend, tell me quickly the numbers, (which when) divided (separately) by 4, 5, 8, or 9, either leave no remainder or leave 1 as the remainder or leave 21 as the remainder."

Definition

R.36(d)-37(a) "A definition coming in a particular step in the process is to be used in that step of the process."

A SPECIAL PROBLEM

R.37(b)-39 "The argument in 'the rule of three' is the divisor, the number defininig (the constituent) is the dividend (and the remainder is the subtractive). The quotient (obtained) hapens to be the argument) (and) the multiplier is its requisition. The earlier remainder is the multiplier and the still earlier remainder is the earlier (multiplier) and so on. (Finally) for (the integral number of years) and the unknown requisition as well, the fruit in (the rule of) proportion (i.e., the rule of three) is the dividend. The multiplier obtained happens to be the requisition and the quotient its fruit (i.e., fruit of the requisition). The requisition either so obtained or if given, may as well be used to find the fruit of the demand by the rule of) proportion (i.e, by 'the rule of three)."[92]

Explanation and Rationale : Let a man cover a distance of b *yojanas* in d years. With this rate, let him travel a distance of p *yojanas* in q years. Then,

$$d/b = q/p$$

or q (in years) = n years, m months, x days, and $y + (c/b)$ *ghaṭīs* (say).

Now suppose that neither p nor q is known but only the remainder of *ghaṭīs* after division by b (i.e., only c) is known. The rule gives a process to find p and q both. Since 60 *ghaṭīs* make a day, so 60 is made the dividend. Also, the argument (in 'the rule of three') is b, so b is made the divisor and c, the remainder of *ghaṭīs* after division by b, the subtractive. Thus the equation

$$(60x - c)/b = y$$

is formed and solved with known b and c. Now, the quotient y, gives the number of *ghaṭīs* and the multiplier x, the remainder of days (after division by b) as $60x/b = y + (c/b)$, and 60 *ghaṭīs* make a day.

Here, it may be noted that only one solution of the above equation wil be valid as $x < b$ and $y < 60$ restricts the valid solution of the equation. Not only that but at all the successive stages of solution of the indeterminate equations formed at those stage inthe solution of the problem also, only one solution of those equation will be valid due to similar restrictions. Now, making necessary changes in the coefficient of x and in the subtractive quantity and repeating the above process we obtain the number of days, months and remainder of years (after division by b). Finally. the fruit is made the dividend, the remainder of years (after division by b), the subtractive and as usual, the argument, the division. Thus, the equation formed is

$$(dx - c')/b = y,$$

where c' is the remainder of years (when divided by b). So,

$$dx/b = y + (c'/b)$$

Now, the quotient y gives the number years. So $y + (c'/b)$ gives the total time taken in covering the distance. Now, since the man covers a distance of b *yojanas* in d years, so obviously, the multiplier x will give the total distance traversed by the man and hence the rule.

The problem can also be solved by 'the rule of three' if either the distance traversed or the time taken is knwon.

The rule is[93] extensively used in Hindu astronomy for finding (the place of) a planet and the elapsed days, from the remainder of *ghaṭīs* in the planets place.

Ex.36 "A lame crawls slowly (at the rate of) 61 *yojanas* in 53 years. If the remainder of *ghaṭīs* (after division by 61) be 20, O elder, tell quickly the measure of time in years, months, days and *ghaṭīs*, and the distance traversed."

Solution :Here, 60 *ghaṭīs* = 1 day, the remainder of *ghaṭī* (after division by 61) = 20 and the argument = 61.

So, the equation formed will be,

$$(60x - 20)/61 = y$$

Its general solution by R. 19-25 will be,

the multiplier (= x) = 41 + 61t and the quotient (= y) = 40 + 60t,

where t is any integer. However, since, $x < 61$ and $y < 60$, so its only valid solution will be,

x (= 41) = remainder of days after division by 61 and y (= 40) = number of *ghaṭīs*.

Now, since 30 days = 1 month and the remainder of days = 41, so the next equation formed will be, $(30x - 41)/61 = y$

Its solution will be, $x = 40 + 61m$ and $y = 19 + 30m$, where m is any integer. But as $x < 61$ and $y < 30$ so its only valid solution will be, x (= 40) = remainder of months after division by 61 and y (= 19) = number of days.

Now, since 12 months = 1 year and remainder of months = 40, so the next equation formed will be, $(12x - 40) / 61 = y$

As $y < 12$, its only valid solution will be, x (= 44) = remainder of years after division by 61, and y (= 8) = number of months.

Finally, since the fruit = 53 and the remainder of years = 44, so the equation formed will be, $(53x - 44)/ 61 = y$.

As $x < 61$, its only valid solution will be

x (= the distance traversed) = 25 and y (= the number of years) = 21.

Thus the distance traversed = 25 *yojanas* and the time taken = 21 years 8 months 19 days and 40 + (20/61) *ghaṭīs*.

The chapter on pulveriser in both the books *Gaṇita Kaumudī* and *Bījagaṇitāvataṁsa* (= Bija) by Nārāyaṇa Paṇḍita is[94] nearly the same. However, the following 5 paragraphs (including 2 rules, an example, its solution and a comment, translated below) available[95] in the latter book is not available in the former one.

PULVERISER, WHEN THE SOLUTION IS FRACTIONAL

R.68. (*Bīja*) "Multiply the desired quotient by the divisor. Subtract the additive (from the product). Divide (the difference) by the dividend. (The quotient) happens to be the multiplier in the reduced or unreduced. (both types of) pulveriser."[96]

Ex.36(*Bīja*) "A number is multiplied by 30. Ten is added to the product. The sum is exactly divisible by 12.) expert in pulveriser, tell that."[97]

In this case, the dividend = 30, the additive=10 and the divisor = 12 and so, the equation formed will be $30x + 10 = 12y$.

Here, the dividend and the divisor is divisble by 3, but the additive is not divisible by 3. So, according to R.19 the problem is incorrect. However, according to Nārāyaṇa,[98] if the quotients are taken as 1, 2 and 3, the multipliers are obtained as 1/15, 7/15, and 13/15 and so on in many ways, according as the optional numbers are taken.

R.69.(*Bīja*) "In a pulveriser, the absence of the process (of pulveriser), with integral quotient, the multiplier obtained is not always (so). While obtaining the solution of (such) a pulveriser, due to the desired number, the multiplier obtained is a fraction."[99]

At the end of the chapter on pulveriser in the *Bījagaṇitāvataṁsa*, Nārāyaṇa comments,[100] "Certain skills in pulveriser have been told ahead in (the chapter on) simultaneous linear equations." Though a part of Nārāyaṇa's *Bījagaṇitāvataṁsa* is available, probably its chapter on simultaneous linear equations (*anekavarṇasamīkaraṇa*) is now lost.

REFERENCES FOR CHAPTERS V TO IX

1. Singh. P., The *Gaṇita Kaumudī* of Nārāyaṇa Paṇdita, *GB*, 20(1998), 25-82, and 21(1999), 10-73, contain translations of earlier chapters I to IV.

2-3. Datta B.B. and Singh A.N. Hindu Geometry, *IJHS*, Vol. 15(2), 1980, 175-176

4-5. Gupta, R.C., Brahmagupta's Rule for the Volume of Frustum-like Solids, M.E., Vol. VI(4), Dec. 12972, 117-120.

6-7. Refer, 3, 175-176.

8-10 Refer, 5, 117-120.

11. For details, refer 8.

12. Cf. *Tri*, vss. 52-53; *MS*, XV, vss. 104-105; *SS'*, 64; *L*, 128,

13. Cf. *BSS*, Xii, vss. 45-46; *GSS*, 253; *MS*, XV, vs. 106; *SS'*, 64; *L*, 128-131.

14. Sarasvati Amma, Dr. T.A., *Geometry in Ancient and Medieval India*, Moti Lal Banarasi Das, Delhi, 1979, 197.

15. Refer 7, 174.

16. Cf. *MS*, xv, vss. 105-106; *SS'*, 64; *L*, 129;

17. *GK* (II), 199.

18. Cf. *Tri*, vs. 55. See also, *MS*, xv, vs. 107; *SS'*, 68.

19. CF. *L*, 117-118.

20. Gupta, R.C., The process of averaging in Ancient and Medieval Mathematics, *GB*, Vol. 3, Nos. 1-2 (1981), 35.

21. For further details.
(a) See, Gupta, R.C., Bhāskara II's Derivation for the Surface of a Sphere, M.E., Vol. VII (2), (June, 1973) section (B), 50-51. (b) also refer 14, 211-213.

22. For further details, refer 14, 213.
23. For details, refer 7, 180.
24. Refer 21(a), 50-51.
25. Refer 3, 180-181.
26-27. Refer 14, 213.
28. Cf. Tri. vs. 58; *MS*, xv, vs. 109; *L*, 132.
29. Cf. *GSS*, 262; Tri. vs. 58; *MS*, xv, vs. 109; *SS'*, 70; *L*, 132.
30. Cf. Tri.vs. 59-60; MS, xv, vs. 112; *SS'*, 72; L, 134.
31. Cf. *L*,. 135
32. *MS*, xv, vss. 113-114.
33. Cf. Tri.vs. 61; *MS*, xv, vs. 115; *SS'*, 74; *L*, 136-137.
34. Refer 1, 28.
35. See also, Tri. vss. 63-64; *SS'*, 77; *L*, 137-138.
36-37. Refer 7, 183.
38-40. Refer 14, 251.
41. CF. *GSS*, 270-271, See also, *BSS*, XII, vs. 52; Tri. vs. 65; *MS*, xv, vs. 118; *SS'*, 79.
42. *GSS*, 271.
43. Refer 14, 257.
44. See also, *Tri.* vs. 65; MS, xv xv, vss. 118-119, *SS'*, 79.
45. Cf. *Ā*, 57; *BSS*, xii, vs. 53; *SS'*, 81-82; *L*, 141. See also, *GSS*, 278.
46. CF., *L*, 142. See also, *GSS*, 280.
47. Cf., *GSS*, 279; *SS'*, 82; *L*, 142.
48. Cf., Ā, 57-58; *BSS*, XII, vs. 54; *L*, 143.
49. Gupta, R.C.: A Height and Distance Problem from the *Āryabhaṭīya*, *ME*, Vol. VIII, No. 4 (Dec. 1974), section B, 71-75.
50. Refer 1, 34.
51. *HMM* (II), 89-90. However, see also *LBh* (G), 103.
52-54. Cf. (II), 89-90.
55-56. Cf. *BSS*, XVIII, vs. 9; *MBh*, 29; *SS'*, 118, 122; *L*, 146-147; *BBi*, 17-18. See also *LBh*, (G), 105-108.
57-59. Cf. L, 147-148; *BBi*, 18. See also *LBg*(G), 106-108.
60. Cf. *MS*, XVIII, vs.20; *L*,159; *BBi*, 20.
61-63. For details, see *HMM* (II), 96-98.
64. *Ā*, 77-84; *HHM*, (II), 96-99.
65. *MBh*, 29-34; *HHM* (II), 99-100.
66. *SS'*, 118; *HHM*(II), 110.
67. *MBh*, 34-35; *HHM*(II), 101.
68. *BSS*, XVIII, vss. 3-5; and 13; *HHM*(II), 101-103.
69. *GSS*, 115-118; *HHM*(II), 103.
70. *GSS*, 124-126, *HHM*(II), 103-104.
71. Cf. *MS*, XVIII, vs. 20; *L*, 159; *BBi*, 20; See also, *LBh* (G), 109; *GK*(II), 217 (lines 9-10).

72. Cf. *L*, 155-156; *BBi*, 21; *GK*(II), Solution of Ex. 23, 216-217.
73. Cf. *L*, 151-154; *BBi*, 19. See also, *MS*, XVIII, vss. 1-14 and *HHM*(II), 104-108.
74. Cf. *L*, 155-156; *BBi*, 19.
75. Cf. *MBh*, 34-36; *HHM*, (II), 101.
76. Cf. *BBi*, 19. See also, *BSS*, XVIII, vs 13 and *HHM*(II), 121.
77. Cf. L, 157, *BBi*-20.
78. Cf. *MBh*, 32-33; *L*, 159-160; *BBi*, 21
See also, *BSS*, XVIII, vss. 9-11; *MS*, XVIII, vss. 1-14, *HHM*(II), 118-120 and 106.
79-80. Ā, 74-77.
81. *BSS*, XVIII, vs. 5.
82-84. *HHM*, 133-135.
85-87. Shukla, Dr. K.S., Hindu mathematics in the 7th century as found in Bhāskara I's Commentary on the *Āryabhaṭīya* (II), *Gaṇita*, Vol. 23, (june (1972), No. 3, 58-59.
88. *GSS*, 115-118. See also, *HHM* (II), 137.
89. *SS'*, 123.
90-91. *HHM*(II), 139-140.
92. Cf. *L*, 160-164.
93. Refer 87, 60-63; *MBh*, 37-38 and 41-42; *LBh*, 113-114. See also, *L*, 162; *LBh*(G), 110-114.
94. *Bija*, 29-35.
95-100. *Bija*, 35.

CHAPTER - X

INDETERMINATE EQUATION OF THE SECOND DEGREE

(i.e. $Nx^2 \pm c = y^2$)

Introduction

Tenth chapter of the *Gaṇita Kaumudī* deals iwth the indeterminate equation of the second degree (*Varga-Prakṛti*) i.e., $Nx^2 \pm k = y^2$. In it first of all an obvious solution of the equation by supposition has been given in R.1. After that, the Additive Composition has been dealt with in R.2-4(ai) and the Subtractive Composition in R.4(aii)-5(a). The case when the interpolator is the product of a square with an integer has been treated in R.5(b)-6(a). In R.6(b)-7 (a), a method for obtaining a rational solution of the equation has been given R.7(b) gives a method to obtain an infinite number of solutions from one such solution of the equation. Cyclic Method (*Cakravāla*) of solution of the equation has been dealth with in R.8-11. Solutions of the equations $Nx^2 - k^2 = y^2$, $Mn^2x^2 \pm c = y^2$ and $m^2x^2 \pm c = y^2$, have been given in R.12, R.13 and R.14, respectively. a rule to find an infinite numberof solutions of $Nx^2 \pm c = y^2$, with the help of one of its solution and a solution of $Nx^2 + 1 = y^2$, have been given in R.15-16. Approximate value of a quadratic surd is the topic of treatment in R.17.

The indeterminate quadratic equation $Nx^2 \pm k = y^2$,

is called *Varga-Prakṛti* or *Kṛti-Prakṛti* by the Hindus. Here, N is called *Prakṛti*, *Guṇa* and *Guṇaka* (the multiplier) etc. At times x^2 is also called *Prakṛti*. An optional number, taken as the value of (x) is called *Hrasva-mūla*, *Kaniṣṭha-pada* (the lesser root) or *Ādya-mūla* (the first root) etc. A number which when added to or subtracted from the product of 'the square of the lesser root' multiplied by the multiplier (*prakṛti*) yields a number giving a square-root is called *Kṣepaka*, Prakṣepaka, *Prakṣepa*, *Kṣepa* and *Kṣipti* (the interpolator) etc. When negative, an interpolator is also called *Śodhaka* (the subtractive).

Brahmguptą (628 A.D.), Jayadeva (early 11th century A.D. or earlier), Śrīpati (1039 A.D.), Bhāskara II (1150 A.D.) and Nārayaṇa (1356 A.D.) etc. have given rules for the solution of such equations.

SOLUTION OF $Nx^2 + k = y^2$

R.1 "An optionally chosen number is the lesser root. Multiply its square by the multiplier. Add (the interpolator) to or subtract (it) from (the product). (The result) is the square, whose root is the greater root."[1]

The rule is obvious.

BHĀVANĀ; SAMĀSA BHĀVANĀ AND *ANTARA BHĀVANĀ*

R.2-4a(i) "Set down the lesser root, the greater root (and) the interpolator. Set down below them the same set or another set (of similar quantities), in order. Such placing is called *Bhāvanā*. The sum of the two cross-products of the two lesser and the two greater roots happens to be a lesser root. Multiply the product of the two lesser roots by the *prakṛti*. (The product so obtained), added to the product of two greater roots, is a greater root. (In that equation), the product of two interpolators is the interpolator."[2]

R.4a(ii)-5(a) "Alternatively, the difference of the two cross-products (of the two lesser and the two greater roots) is a lesser root. Multiply the product of the two lesser roots by the *prakṛti*. The difference of the products, so obtained) and the product of two greater roots is the greater root. (Again), the product of the two interpolators, is the interpolator."[3]

Explanation : Let $x = m, y = n$ satisfy the equation $Nx^2 + k = y^2$ ...(1)

and $x = m_1, y = n_1$ satisfy the equation, $Nx^2 + k' = y^2$ ...(2)

Then, according to R.2-4a(i), $x = (mn_1 + m_1n), y = (nn_1 + Nmm_1)$ will satisfy the equation, $Nx^2 + kk' = y^2$ ...(3)

and according to R.4a(ii)-5(a), $x = (mn_1 - m_1n), y = (nn_1 - Nmm_1)$ will also satisfy this equation.

In particular, let $m = m_1$ and $n = n_1$, then $x = 2mn, y = n^2 + Nm^2$ will satisfy the equation $Nx^2 + k^2 = y^2$.

Ratrionale : Since $x = m$ and $y = n$ satisfy (1) so, $Nm^2 + k = n^2$ ....(4) and

$x = m_1$ and $y = n_1$ satisfy (2) so, $Nm_1^2 + k' = n_1^2$ ...(5)

From (4) and (5),

$kk' = (n^2 - Nm^2)(n_1^2 - Nm_1^2) = n^2n_1^2 + Nm^2m_1^2 - N(m^2n_1^2 + m_1^2n^2)$

$= (nn_1 \pm Nmm_1)^2 - N(mn_1 \pm m_1n)^2$ Or $N(mn_1 \pm m_1n)^2 + kk' = (nn_1 \pm Nmm_1)^2$

and so, $x = mn_1 + m_1n$, and $y = nn_1 + Nmm_1$ satisfy (3).

The case when $m = m_1$ and $n = n_1$ easily follows from above.

Rules 2-5a are called by the technical name *Bhāvanā* which means demonstration, proof, theorem, lemma, composition and combination etc. These are of two types, *Samāsa Bhāvanā* (R.2-4a(i)), Addition Lemma or Additive Composition, and *Antara Bhāvanā* (R.4a-(ii)-5a), Subtraction Lemma or Subtractive Composition. Subtraction is of two types, *Tylya Bhāvanā* (Composition of Equals) and *Atulya Bhāvnā* (Composition of Unequals). *Tulya Bhāvanā* has been named Brahmagupta's Corollary by B.B. Datta and A.N. Singh. Both the rules R.2-4(ai) and R. 4(aii)-5a, were first of all given by Brahmagupta.

ANOTHER LEMMA

R.5b-6a "(If) the interpolator (of a Square-nature) divided or multiplied by the square of an optional number, be the interpolator (of another Square-nature), (then) the two roots (of the former) divided or multiplied (as the case may be) by that optional number, are the roots (of the other Square-nature)."[5]

According to the rule, if $x = m$ and $y = n$ satisfy the equation,

$Nx^2 \pm p^2K = y^2$ or, $Nx^2 \pm (K/p^2) = y^2$ ....(1)

then $x = m/p, y = n/p$ or $x = pm, y = pn$ will satisfy the equation

$Nx^2 \pm K = y^2$ ....(2)

Rational : Since $x = n$ and $y = n$ satisfy (1), so

$Nm^2 \pm p^2K = y^2$ or $N(m/p)^2 \pm K = (n/p)^2$

and therefore $x = m/p$ and $y = n/p$ will satisfy (2) and similarly the other part can be established.

RATIONAL SOLUTION

R.6(b)-7(a) "Twice an optional number, divided by the difference between the square of that optional number and the multiplier, happens to be the lesser root. From that with unity as the additive, the greater root should be obtained."[6]

Explanation and Rationale : If K be any optional number, according to the rule

$x = 2K/(K^2 \sim N)$ is a root of $Nx^2 + 1 = y^2$ . . . (1)

Obviously then, $y = (K^2 + N)/(K^2 \sim N)$

Since $x = 2K/(K^2 \sim N)$ satisfies (1), so, from (1), we have

$$Y^2 = \left(\frac{2K}{(K^2 \sim N)}\right)^2 + 1 = \left(\frac{K^2+N}{K^2 \sim N}\right)^2 \text{. Therefore,}$$

$x = 2K/(K^2 \sim N)$ and $y = (K^2 + N)/(K^2 \sim N)$

The rule was first of all given by *Śrīpati*. *Bhāskara* II has also given the rule. In Europe, it was[7] rediscovered by Brouncker (1657 A.D.).

INFINITE NUMBER OF SOLUTIONS

R.7(b). "By the Principle of Composition of equal set, as well as unequal set, of roots (will be obtained) an infinite number of roots."[8]

Rationale : In order to obtain an infinite set of solutions of the Square-nature it is necessary to obtain only one solution of it. If (a, b) is solution of the equation, by Brahmagupta's Corollary, another solution of it will be $(2ab, b^2 + Na^2)$. After obtaining 2 solutions of it an infinite number of other solutions of it can be obtained by repeated use of the Principle of Composition (Rules 2–5*a*).

In the west Fermat (1657 A.D.) was the first to claim that the equation [9] $Nx^2+1=y^2$, N being a non-square integer, has an infinite number of solutions in integers. In India Śrīpati, Bhāskara II and Nārāyaṇa clearly mentioned the existence of an infinite number of integral solutions of the equation, long before Fermat.

Ex.1 "O friend, tell me quickly (the number) whose square (when) multiplied by 8 and then added to 1 yields a square-root or thinking well (tell the number) whose square if multiplied by 11 (and then) added to 1 becomes square."

Solution : Case1 : Here the equation is, $8x^2 + 1 = y^2$ . . . (1)

Let the lesser root (= *h*) be 1. Then the greater root (= *j*) for the interpolator (=*k*) 1 will be 3. Now the statement for composition is,

$N = 8$ $h = 1$ $j = 3$ $k = 1$ $h = 1$ $j = 3$ $k = 1$

By the Principle of Composition of Equals, we get,

$N = 8$, $h(= 2.1.3) = 6$, $j(=8.1^2 + 3^2) = 17$, $k(=1.1) = 1$, R.2-4a(i).

Statement for Composition for the next case will be,

$N = 8,$ $h = 1$ $j = 3$ $k = 1$ $h = 6$ $j = 17$ $k = 1$

Now, by the principle of Composition of Unequals (R.4(aii)–5a), we get, $N = 8$, $h(= 1.17+3.6) = 35$, $j(=8.1.6+3.17) = 99$ when $k(= 1.1) = 1$.

Again, statement for Composition for the next case will be,

$N = 8$ $h = 1$ $j = 3$ $k = 1$ $h = 35$ $j = 99$ $k = 1$

Now, by the Principle of Composition of Unequals, we get, $N = 8$, h $(=3.35+1.99) = 204$, $j(=8.1.35+3.99) = 577$ when $k (= 1.1) = 1$ and so on and "thus an infinite number of roots" may be found. Alternatively, take the equation $8x^2 + 4 = y^2$ Here, $h = 2, j = 6$ when $k = 4$.

Now by R.5b – 6a, we get, $h = 1, j = 3$ when $k = 1$ as the solution of the equation $8x^2 + 1 = y^2$

Now, again an infinity of solutions of the latter equation can be obtained by the Princile of Composition of Equals as well as Unequals. Again, by R.6(b)-7(a), if the optional number $(= m) = 3$; $h = 6, j = 17$ for $k = 1$ and if, $m = 5$; $h=10/17, j = 33/17$, for $k = 1$.

Now from these two solutions, by Additive Composition, we get,

$h = 368/17, j = 1041/17, k = 1$ and by Subtractive Composition, $h = 28/17$, $j = 81/17, k = 1$ and so on.

Nārāyaṇa states. "With unity as the interpolator an infinity of roots are obtained by the use of the Principle of Composition of Equals as well as Unequals."

Case 2 : Here the equation is, $11x^2 + 1 = y^2$ . . . . (1)

Consider the equation $11x^2 - 2 = y^2$

We have, $N = 11$ $h = 1$ $j = 3$ when $k = -2$.

Statement for Composition will be

$N = 11$ $h = 1$ $j = 3$ $k = -2$ $h = 1$ $j = 3$ $k = -2$

Now by the Principle of Composition of Equals, $N = 11$, $h(=2.1.3) = 6$, $j(=11.1.1+3) = 20$ and $k = (-2)(-2) = 4$. R.5b–6a, now gives, $N = 11$, $h = 3$, $j=10$ for $k = 1$

Again by the Principle of Composition of Equals, we get, $N = 11$, $h = 60$, $j = 199$ for $k = 1$

Alternatively, take the equation as $11x^2 + 25 = y^2$

We have, $N = 11$, $h = 8$, $j = 27$ for $k = 25$

By R.5b-6a, we get, $h = 8/5$, $j = 27/5$ for $k = 1$.

Also from above, $h = 3, j = 10$ for $k = 1$.

From Additive Composition, we now get $N = 11$, $h = 161/5$, $j = 534/5$, for $k = 1$ and again "an infinite number of roots" of (1) may be obtained. Also, from Subtractive Composition we get $N = 11$, $h = 1/5$, $j = 6/5$, for $k = 1$ and again "an infinity" of solutions of (1) can be obtained. Again, by R6(b)-7(a), if the optional number (=m) = 3;

$$h = 3, j = 10 \text{ for } k = 1 \text{ and if } m = 5;\ h = 5/7, j = 18/7 \text{ for } k = 1.$$

Now from these two solutions, by Additive Composition, we get, $h = 104/7$, $j = 345/7$, $k = 1$ and by Subtractive Composition, $h = 4/7$, $j = 15/7$, $k = 1$ and so on and "thus an infinite number of roots" may be obtained.

Cakravāla (Cyclic Method)

R.8-11 "Making the lesser root, the greater root and the interpolator (of a Square-nature), the dividend, the addend and the divisor, (respectively) of a pulveriser, determine the indeterminate multiplier of it in the way described before. Subtract the *prakṛti* from the square of the multiplier or the square of that from the *prakṛti*. The remainder divided by the (original) interpolator is the interpolator (of a new) Square-nature and reverse its sign in case of subtraction of the square of the multiplier. The quotient (corresponding to that value of the multiplier) is the lesser root (of the new Sqaure-nature) and that multiplied by the multiplier and diminished by the product of the previous lesser root and new interpolator is its greater root. By doing so repeatedly, two integral roots corresponding to the interpolator ± 1, ± 2 or ± 4 are obtained. Integral roots for the additive unity may be obtained from those corresponding to the interpolator (-1), ± 2 or ± 4 by the Principle of Composition."[10]

Explanation : To solve, $Nx^2 + 1 = y^2$

first of all an auxiliary equation $Na^2 + k = b^2$ . . . (1)

is formed, in which a, k, b are known and N is known from (1). Now a pulveriser, $(am + b)$ = an integer = p, is formed and general value of m is . . . (2) obtained from it. A suitable value of $m(= m_1)$ preferably, though not necessarily,[11] that value which makes $m^2 - N$, the least) is taken and another auxiliary equation $Na_1^2 + k_1 = b_1^2$

is formed such that $(am_1 + b)/k = a_1$, $(m_1^2 \sim N)/k = k_1$ and $a_1m_1 - k_1a = b_1$. . . . (3)

Now the process is repeated with (3) in place of (2) and a similar auxiliary equation $Na_2^2 + k_2 = b_2^2$ is obtained. Repetition of the process, again and again, leads to an auxiliary equation in which the interplator is ± 1, ± 2 or ± 4 which gives the solution of equation (1) by the use of the Principle of Composition.

In the words of C. Selenius,[12] the *cakravāla* method represents a best approximation algorithm of minimum length that owing to several minimization properties with minimum effort and avoiding large numbers always (without trial

process) produces the least solution to the equation and thereby the whole set of solutions.

Rationale : (a) *Nārāyaṇa's Lemma* The Cyclic Method of Nārāyaṇa is based upon the following Lemma : If $Na^2 + k = b^2$ where a, b, k are integers, k being positive or negative, then

$$Na_1^2 + k_1 = b_1^2 \qquad \ldots (3)$$

where,

$a_1 = (an + b)/k$ . . . (4) $k_1 = (n^2 - N)/k$ . . . (5) and $b_1 = a_1n - k_1a$ . . . (6)

We have, $Na^2 + k = b^2$. Also $N.\ 1^2 + (n^2 - N) = n^2$, identically. Now by the application of Additive Composition to these two relations, we get

$N(an + b)^2 + k(n^2 - N) = (Na + bn)^2$ or $N(an + b)/k^2 + (n^2 - N)/k = (Na + bn)/k^2$ . . .(7)

$$\text{But, } \frac{Na + bn}{k} = \frac{n(a_1k - an) + Na}{k} = a_1n - \left(\frac{n^2 - N}{k}\right)a \text{ from (4)}$$

$(= a_1n - k_1a) = b_1$, from (5) and (6). So, from (7), we get, (3), due to (4), (5) and (6).

b) *k_1, b_1 are integral if a_1 is integral*. From $Na_1^2 + k_1 = b_1^2$, it is clear that b_1 will be integral if k_1 is so, as N and a_1 are integral . From above, we have,

$b_1 = (Na + bn)/k$ and $a_1 = (an + b)/k$ from (4)

Eliminating b between these two relations, we get,

$k(a_1n - b_1) = a(n^2 - N)$ or $k(a_1n - b_1)/a = n^2 - N$. Therefore $k(a_1n - b_1)/a$ is an integer. Now, k and a have no common factor, so a must divide $a_1n - b_1$.

Therefore, $(a_1n - b_1)/a = (n^2 - N)/k = k_1$, an integer.[13]

(c) *Recursive Character* : The *cakravāla* process of solving $Nx^2 + 1 = y^2$, has intimate relation with the continued fraction. Several authors such as, Ayyanger,[14] A.A.K. and Majumdar[15], P.K. etc. have treated the topic. Selenius,[16] C. and the present[17] author have explained the process (including its recursive character), the former with the help of semi-regular continued fraction and the latter with the help of regular continued fraction. Since we will need it again in explaining the next rule, let us define it here and state some of the results associated with it.

Let a_n, b_n and r_n be defined[18] by the relations,

$$\frac{\sqrt{N} + b_n}{r_n} = a_n + \frac{\sqrt{N} - b_{n+1}}{r_n} = a_n + \frac{r_{n+1}}{\sqrt{N} + b_{n+1}}$$

for values of n = 1, 2, 3, ... , where an is an integer such that,

$$a_n \langle \left[\left(\sqrt{N}+b_n\right)/r_n\right] \langle a_{n-1}$$

$b_{n+1} = a_n r_n - b_n$ and $r_n r_{n+1} = N - b^2_{n+1}$

leading to,

$$\sqrt{N} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_{c-1} +} \ \frac{1}{a_c +} \ \frac{1}{2a_1 +} \ \frac{1}{a_2 +} \cdots$$

so that the number of elements in the recurring cycle is c. Let p_r/q_r be rth convergent for the continued fraction.

The whole process[19] of *cakravāala* has been explained with the help of regular continued fraction. It has[20] been shown that all quantities in the *cakravāla* process have simple counter parts in the regular continued fraction expansion of $\sqrt{N}$. Thus, r_n is[21] the interpolator, q_n, the lesser root and p_n, the greater root. moreover, not only the recursive[22] charater of the *cakravāla* process has been proved which at once establishes that the process will always lead to the solution of (1), but Principle of Composition and Brahmagupta's another Lemma have also been explained in that light.

(d) *Principle of Composition* Though the *cakravāla* process always leads to the solution of $Nx^2 + 1 = y^2$ . . . (1)

due to its recursive character, the use of the Principle of Composition at a stage when any one out of – 1, ± 2, ± 4 is obtained as an interpolator, brilliantly shortens the proces..

(i) If $k = -4$, use of Brahmagupta's Corollary will immediately lead to a relation with 4 as the additive as the use of the corollarly will make the lesser root (and so the greater root as well) divisible by 4.

(ii) If $k = 4$ and the roots are even, use of Brahmagupta's Corollary will immediately lead to a solution of (1) as it will make the lesser root (and so the greater root as well) divisible by the interpolator16.

(iii) $k = 4$ and the roots are odd. Let the relation be $Na^2 + 4 = b^2$ . . .(2)

By Brahmagupt's Corollary, we obtain,

$N(2ab)^2 + 4.4\ [= (b^2 + Na^2)^2] = [2(b^2 - 2)]$, due to (2); or, $N(ab)^2 + 4 = (b^2 - 2)^2$ . . .(3)

By the Principle of Composition of (2) and (3), we have

$N\,[a(b^2 - 2) + ab^2]^2 + 4.4 = [b(b^2 - 2) + Na^2b]^2$, or $N[2a(b^2 - 1)]^2 + 4.4$

$= [2b(b^2 - 3)]^2$, due to (2).

This gives the solution of (1) as a and b both are odd and so both the sides are divisible by 16.

(iv) If $k = \pm 2$ Brahmagupta's Corollary will immediately lead to the solution of (1) due to the interpolator will now become 1.

(v) If $k = -1$, Brahmagupta's Cor will lead to the solution of (1).

Earliest treatment of *cakravāla* method for the solution of *varga prakṛti*[23] is by Ācārya Jayadeva (early 11th century A.D.). The treatment is found in the form of quotations from his work in the *Sundarī* commentary (1073 A.D.) by Udayadivākara on the *Laghu - Bhāskarīya* of Bhāskara I. According to C.Selenius, *cakravāla* method is considered[24] to be the absolute climax of the old Indian mathematics and so of all Oriental mathematics. This method anticipated[25] the European methods by more than thousand years. No European performances in the whole field of algebra at a time much later than Bhāskara's, may upto our time, equalled the marvellous complexity and ingenuity of *cakravāla*.

Ex.2 "Tell me quickly that (number) which (when) multiplied by 103, (the product) added to 1, becomes a square or tell (the number) which (whn) multiplied by 97, the product added to 1, becomes a square."

Solution : *Case I.* The equation is $103x^2 + 1 = y^2$ . . . (1)

Let the auxiliary equation be $103.(1)^2 - 3 = (10)^2$

Here, the lesser root (=a)=1, the interpolator (=k)= – 3 and the greater root (=b)=10. So, the pulveriser formed (R.8) will be $(1.m + 10)/(-3)$ (= *an* integer) = s

The general solution of this will be $m = -3p + 2$ where p is any integer. Taking $p = -3$, we have, $m = 11$. Now, the next auxiliary equation formed (R.8-10) will be $103a_1^2 + k_1 = b_1^2$,

where, $a_1\left(= \frac{m+10}{-3}\right) = -7$. $\quad k_1\left(= \frac{m^2 - 103}{-3}\right) = -6$ and $b_1(= a_1 m - k_1 a) = -71$. So, the next auxiliary equation is, $103(-7)^2 - 6 = (-71)^2$.

Now, according to Nārāyaṇa[26], "For further working there is no difference between negative and positive roots". So taking the roots as positive, the auxiliary equation becomes $103(7)^2 - 6 = (71)^2$.

The next pulveriser will be $(7n + 71)/(-6)$ = an integer = t.

Its solution will be $n = -6q + 1$, where q is any integer. Taking $q = -1$, we obtain, $n = 7$. Now the auxiliary equation, $103a_2^2 + k_2 = b_2^2$ will be

$103(20)^2 + 9 = (203)^2$, as $a_2\left(= \frac{7.7 + 71}{-6}\right) = -20$, $k_2\left(= \frac{7^2 - 103}{-6}\right) = 9$

and $b_2\,(= a_2 n - n - k_2 a_1) = -203$.

The next pulveriser will be $(20v + 203)/9$ = an integer = r.

Its solution will be $v = 9w + 2$. Taking $w = 1$, we get, $v = 11$.

As done above, this leads to the auxiliary equation $103(47)^2 + 2 = (477)^2$

Now by Brahmgupta's Corollary,

$103(2.47.477)^2 + 2^2 = (477^2 + 103.\ 47^2)^2$ or, $103(22419)^2 + 1 = (227528)^2$, after simplification.

This gives $x = 22419$, $y = 227528$, as the solution of (1)

Case II, Here, the equation is, $97x^2 + 1 = y^2$ ...(1)

Let the auxiliary equation be, $97(1)^2 + 3 = (10)^2$

The pulveriser formed (R.8) will be, $(1.m + 10)/3$ = an integer = p,

as the lesser root = 1, the greater root = 10 and the additive = 3. Its solution will be $m = 3h + 2$ where h is any integer. Taking $h = 3$, we have $m = 11$. Now the next auxiliary equation will be (R.8-10) $97a_1^2 + k_1 = b_1^2$,

where, $a_1 \left(= \frac{1.m + 10}{3}\right) = 7$, $k_1\left(= \frac{m^2 - 97}{3}\right) = 8$ and $b_1 (= a_1 m - k_1 a) = 69$. So, the equation becomes, $97(7)^2 + 8 = (69)^2$

The next pulveriser formed (R.8) will be, $(7.n + 69)/8$ = an integer = q.

Its solution will be $n = 8b + 5$, where b is any integer. Taking $b = 1$, we obtain, $n = 13$. So, the next auxiliary equation formed (R.8-10) will be, $97(20)^2 + 9 = (197)^2$

Proceeding in a similar manner, the next pulveriser will be,

$(20r + 197)/9 = w$.

Its solution will be, $r = 9c + 5$. Taking $c = 1$, we get, $r = 14$.

We have noted earlier that in taking the value of the multiplier (= r), Nārāyaṇa does not expressly state that r should be so chosen as will make $|r^2 - N|$ the least. Here, we note that $r = 5$ will make $|5^2 - 97|$, the least and not $|14^2 - 97|$, but Nārāyaṇa has[27] taken $r = 14$ and not $r = 5$.

The next auxiliary equation will be, $97(53)^2 + 11 = (522)^2$

The next pulveriser will be, $(53s + 522)/11 = u$.

Its solution will be $s = 11d + 8$. Taking $d = 0$, we have, $s = 8$. Now, the next auxiliary equation will be, $97(86)^2 - 3 = (847)^2$.

The next pulveriser wil be $(86t + 847)/(-3) = v$.

Its solution will be, $t = -3e + 1$. Taking $e = -3$, we obtain, $t = 0$ With this value of v,

With this value of v, the next auxiliary equation will be, $97(569)^2 - 1 = (5604)^2$

As, −1 has been obtained as interpolator, by Brahmagupta's Corollary, we obtain

$97(2.569.5604)^2 + 1 = (5604^2 + 97.569^2)^2$ or $97(6377352)^2 + 1 = (62809633)^2$

Thus, $x = 6377352$, and $y = 62809633$ is a solution of (1).

SOLUTION OF $Nx^2 - k^2 = y^2$

R.12 "In case of unity as the subtractive, the multiplier is the sum of two squares. If not, the solution is impossible. If the multiplier be such that the solution is possible, the greater and the lesser roots should be obtained by the method stated earlier."

Explanation and Rationale : According to the rule, (a) if the equation is,
$Nx^2 - 1 = y^2$ ...(1)

N must be the sum of two squares. For, if $x = p/q$ and $y = r/s$, satisfy (1), we have,

$N(p/q)^2 - k^2 = (r/s)^2$ or $N = (rq/ps)^2 + (qk/p)2$

(b) The equation can be solved by *cakravāla* method.

As is clear by the solutions of the examples given by Nārāyaṇa if, $N = m^2 + n^2$, two rational solutions of (1) will be, $x = 1/m$, $y = n/m$ and $x = 1/n$, $y = m/n$ and so, two rational solutions of,

$Nx^2 - k^2 = y^2$ will be, $x = k/m$, $y = kn/m$ and $x = k/n$, $y = km/n$.

According to Nārāyaṇa, "Integral roots (should be obtained) by *cakravāla* (Cyclic Method)." Besides these, Bhāskara II has[28] given several other methods for the solution of such equations.

Ex.3 "Tell, which number (when) multiplied by 13, (the product) lessened by 1, yields a square or which number (when) multiplied by 61, (the product) lessened by 1, becomes a square."

Ex.4 "Which (is the number whose) square multiplied by 5, (the product) either added to 4 or, to 36 or, to 100, happens to be a square."

Solution of $Mn^2x^2 \pm c = y^2$

R.13 "Divide the multiplier (of a Square-nature) by an arbitrary square number, so that, no remainder is left (after division). Take the quotient as the multiplier of another Square-nature. The lesser root (of the reduced equation), divided by the square-root of the divisor, will be the lesser root (of the original equation)."[29]

Let the equation be, $Mn^2x^2 \pm c = y^2$ . . . (1)

so that the multiplier is divisible by n^2. Let another equation, $Mz^2 \pm c = y^2$ be formed such that $z = nx$ and left the lesser root of this equation be $z = a$. According to the rule, the lesser root of (1) will be a/n, the greater roots of both the equations being the same. The rule is obvious.

Ex.5 "O friend, tell quickly (the number whose) square multiplied by 72, (the product) added to 1 happens to be a square, or (tell number whose) square multiplied by 1/5 (the product) added to 1 happens to be a square."

Solution of $m^2x^2 \pm c = y^2$

R.14 "The interpolator, divided by an optional number, is set down at two places. The quotient is diminished (at one place and) increased (at the other) by that (otional number and then) halved. The former is divided by the square-root of the multiplier. These are the lesser and the greater roots, in order."[30]

According to the rule, $x = \frac{1}{2m}\left(\frac{\pm c}{k} - k\right)$, $y = \frac{1}{2}\left(\frac{\pm c}{k} + k\right)$.

where k is an optional number, is the solution of $m^2x^2 \pm c = y^2$ . . . (1)

Rational : The equation (1) can be written as

$$\pm c = y^2 - m^2x^2 = (y - mx)(y + mx) = k(y + mx),$$

where, y

$$y - mx = k.$$

Now,

$y + mx = \pm c/k$. So, by the rule of concurrence, we have

$$x = \frac{1}{2m}\left(\frac{\pm c}{k} - k\right), and \quad y = \frac{1}{2}\left(\frac{\pm c}{k} + k\right)$$

and hence the rule.

Ex.6 "O mathematician, tell me quickly the number whose square, multiplied by 9, the product (either) increased or diminished by 10, is a square."

Ex.7 "O learned, if you know, tell quickly, the number whose square multiplied y 10, (the product) either increased or diminished by 10, happens to be a square."

Ex.8 "If you know Square-nature, tell the number wose square multiplied by –5, (the product) added to 21, happens to be a square."

Solution of $Nx^2 \pm c = y^2$ (Alternative Rule)

R.15-16 "When the additive or the subtractive is greater than unity, the (two) roots should be determined by one's own intelligence. Then by the Principle of Composition of the roots with the additive unity, an infinite number of roots (can be obtained). Persons having no intelligence, have no knowledge of mathematics. They should be told the whole of mathematics by one's own intelligence."[31]

Rationale : Let $x = a, y = b$ be a solution of $Nx^2 + c = y^2$, found empirically, so that, $Na^2 + c = b^2$. Also, let $x = p, y = q$, be a solution of $Nx^2 \pm 1 = y^2$, so that, $Np^2 \pm 1 = q^2$.

By the priniciple of Composition of above relations we get,

$$N(bp \pm aq)^2 \pm c = (bq \pm Nap)^2$$

and so $x = bp \pm aq, y = bp \pm Nap$ will be a solution of, $Nx^2 \pm c = y^2$ . . . (1)

An infinite number of solution of (1) will be obtained by the repeated use of the Principle of Composition.

Ex.9 "O friend, tell me (the number whose) square being multiplied by 13 (and then) increased or diminished by 17, separately, becomes capable of yielding a square-root."

Approximate Value of a Quadratic Surd

Techniques for finding approximate value of a quadratic surd were known[32] to Indians centuries before Christ. Baudhāyana gave a rule which implies,[33]

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}.$$

The same rule is found in the *Śulba Sūtras* [34] of Āpastamba and Kātyāyana. The Berlin Museum Greek Papayrus (A.D.2nd cent) contains[35].

$$\sqrt{164} = 12 + \frac{2}{3} + \frac{1}{15} + \frac{1}{26} + \frac{1}{32}$$

The approximation,

$$\sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{\left(\frac{r}{2a}\right)^2}{2\left(a + \frac{r}{2a}\right)}$$

is[36] found in the Bakṣāli Manuscript. The Jainas approximated[37] π by $\sqrt{10}$. Thus, approximate values for $\sqrt{10}$ have been given [38] in ancient and medieval Jaina works such as the *Anuyogadvāra Sūtra, the Tiloyapaṇṇati* and the *Tiloya Sāra*. The Jainas were[39] able to obtain $\sqrt{10}$ correct to 13 decimal places.

Several other Methods of approximating quadratic surds were [40] prevalent during ancient and medieval periods in cultural areas, as well. Śrīdhara gave [41] a

rule, according to which $\sqrt{N} = \left(\sqrt{NA^2}\right)/A$

The same rule has also been given[42] by Śrīpati Bhāskara II and Nārāyaṇa.

R.17 "(With the number), whose root is to be found out, as the multiplier and unity as the interpolator, (obtain the greater and the lesser) roots. Divide the greater root by the lesser root. The quotient is an approximate value (of the number under the radical sign)."

According to the rule, if $x = a, y = b$ is a solution of $Nx^2 + 1 = y^2 \ldots (1)$

then b/a is an approximate value of $\sqrt{N}$.

As we know, by the Principle of Composition (R.2-55a), we can derive an infinite number of roots of (1) and thus can derive an infinite number of approximate values of $\sqrt{N}$. Each applcation of Brahmagupta's Corollary or Additive Composition will give to $\sqrt{N}$ a value nearer to it. Another interesting aspect of the above method is that at any stage of approximation, we know the upper limit and the lower limit between which the error in taking the approximate value in place of the exact value, lie.

With the notations of the previous rule [R.8-11, Rational (Recursive Character)], as we know, text books on algebra[43] show that the error committed in taking p_n/q_n as the value of the cntinued fraction i.e., ($\sqrt{N}$ expressed as a continued fraction) is less than $\frac{1}{q_n q_{n+1}}$ and greater than $\frac{a_{n+2}}{q_n q_{n+2}}$.

In fact as Garver, R. has shown,[44] the error, in taking $\frac{p_n}{q_n}$ $\left(or \ \frac{p_{2n}}{q_{2n}} \right)$ as the value of $\sqrt{N}$, is greater than $\frac{1}{2p_n q_n} \left(or \ \frac{1}{2p_{2n} q_{2n}} \right)$ but less than $\frac{1}{2q_n^2 \sqrt{N}} \left(or \ \frac{1}{2q_{2n}^2 \sqrt{N}} \right)$ according as c is even (or odd).

Thus if b/a is taken as an approximate value of $\sqrt{N}$, where $x = a, y = b$ is a solution of the equation, $Nx^2 + 1 = y^2$

the error in this approximation[45] is greater than $\frac{1}{2ab}$ and less than $\frac{1}{2a^2\sqrt{N}}$.

Ex.10. "O friend, if you know the process of *varga prakṛti*, tell the approximate root of 10 or 1/5."

Solution :Case I Consider, $10x^2 + 1 = y^2 \quad \ldots (1)$

Taking the optional number as 3, by R.6(b) – 7(a), we find that $x = 6, y = 19$, is a solution of (1). So, 19/6 will be an approximate value of $\sqrt{10}$. Now, by Brahmagupta's Corollary, $x = 228, y = 721$, is another solution of (1). So, 721/228 is another approximate value of $\sqrt{10}$.

Now, by the Additive Composition, from the above two solutions, we find that, $x = 8658, y = 27379$ is another solution of (1). Therefore 27379/8658 is another approximate value of $\sqrt{10}$ and so on. Also, each subsequent value will be nearer to $\sqrt{10}$ in comparison to its preceding approximation. The example is very important as $\sqrt{10}$ was widely used in ancient world for π.

Case II Consider, $(1/5)x^2 + 1 = y^2 \ldots (2)$ Taking the optional number as 1/2, as done[46] by Nārāyaṇa in his *Bījagaṇitāvataṁśa*, we find by R.6(b)-7(a) that $x = 20, y = 9$ is a solution of (2). Use of Brahmagupta's Corollary gives, $x = 360, y = 161$ as the next solution of (2). Again, from the above two solutions, by Additive Composition, we find that $x = 6460, y = 2889$ is another solution of (2).

Therefore, approximate values of $\sqrt{\frac{1}{5}}$ *are* $\frac{9}{20}, \frac{161}{360}, \frac{2889}{6460} \ldots$ and so on.

References for Chapter X

1. Cf. *BBi*, 22.

2-3. Cf. *BŚS*, Chap.XVIII, vss. 64-65; *BBi*, 22-23. See also, *HHM* (I), 146-149.

4. *HHM*(II), 146-149.

5. cf. *BSS*, Chap. XVIII, vs.65.

6. Cf. *SŚ*, 130; *BBi*, 27.

7. *HHM*(II), 155.

8 Cf.*SŚ*, 130; *BBi*, 23.

9. *HHM*(II), 150.

10. Cf. Shukla, K.S., Ācārya Jayadeva, the mathematician, *Gaṇita*, vol. 5, No.1, June 1954, 1-20; *BBi*.22-24.

11(a) See the solution of Ex.2 given later on.

(b) For the reasons and details, see, Singh, Parmanand, *Varga-Prakṛti* - the *Cakravāla* Method of its Solution and the Regular Continued Fractions, *IJHS*, Vol. 19, No.1, 1983, 9.

12. C.Selenius, Rationale of the *cakravāla* process of Jayadeva and Bhāskara II, *Historia Mathematica*, II (1975), 167-184.

13. For further details, see *HHM*(II), 165-166.

14. Refer 12, References. Selenius has given a list of 6 articles by A.A.K. Ayyangar on th topic.

15.(a) Majumdar, P.K., *Gaṇita Kaumudī* and the continued fraction, *IJHS*, Vol. 13, No.1, 1-5.
(b) See also, Majumdar, P.K., Indeterminate Quadratic Equation in Indian and Central Asian Context, *Interaction between Indian and Central Asian Science and Technology in Medieval Times*, Vol.1, 271-277.
16. Refer 12.
17. Refer 11(b).
18. Barnard, S. and Child, J.M, *Higher Algebra*, Macmillan and Col. Ltd., London, 1952, 530-531.
19-22. Refer 11(b), 1-17.
23. Refer 10, 1-2.
24-25. Refer 12.
26. GK(II), 237.
27. Refer 11(b), 9.
28. Cf. *BBi*. 24-25. See also *HHM*(II), 175-181.
29. Cf. *BSS*. Chap. XVIII, 70; *BBi*. 25. See also, *HHM*(II), 175-176
30. Cf. *BSS*. Chap. XVIII, vs.69; *BBi*, 25. See also, *HHM*(II), 176-177
31. Cf. *BSS*, Chap. XVIII, vs.66; *BBi*, 25. See also, *HHM*(II), 173-175.
32. Channabasappa, M.N., On the Square Root Formula in the *Bakṣali Manuscript*, *IJHS*, Vol. 11, No.2., 113.
33. Gupta R.C., Vedic Mathematics from the *Śulba Sūtras*, *Indian Journal of Mathematics Education*, Vol.9, No.2,5.
34. Gupta, R.C., Baudhāyana's Value of $\sqrt{2}$, *ME*, Vol. VI, No.3., Sec. B.,77.
35. Gupta, R.C.; Square Root of 164 in the Berlin Papyrus 11529; *GB*, Vol.2 No. 1-2,29.
36-37. Refer 32, 112-113.
38. Gupta R.C., Circumference of the Jambudvipa in Jaina Cosmography; *IJHS*, Vol.10, No.1, 39-42.
39. Refer 32, 113.
40. Gupta, R.C., On some ancient and medieval methods of Approximating Quadratic Surds, *GB*, Vol. 7., Nos. 1-4, 13-22.
41. *PG*, 91.
42. *SŚ*, 54; *L*, 77-78; *GK*, Part II, 33.
43. Refer 18. 400.
44. Garver, Raymond, Concerning Two Square Root Methods. *Bulletin of the Calcutta Mathematical Society*, 24(2), 1932, 99-102.
45. For details, see, Singh, Parmanand, Nārāyaṇa's Method for evaluating quadratic surds and the regular continued fraction expansions of the surds, *ME*, Vol. XVIII, No.2, 63-65.
46. *Bīja*, 44.

CHAPTER XI

DIVISORS OF A NUMBER

In R.1 Nārāyaṇa introduces the subject. Ordinary method of finding divisors of a number has been given in R.2-4(a) while R.4(b) deals with the case when the number is the difference of two squares has been given in R.5-6(a). A process to convert a number into the difference of two squares, to find its divisors, has been given in R.6(b)-8(a). A method for writting all the divisors of a number has been given in r.8(b)-8(c). A shorter process for finding divisors of a number has been given in R.9-10. Lastly, R.11 gives a rule for testing correctness or otherwise of a multiplication.

Early Hindu works on arithmetic do not treat[1] the topic of factoring (or finding divisors of) a number. However, there are resons to believe[2] that the ordinary method of factoring a number by successive divison by 2, 3 and 5 etc. were well known, However, this being much too elementary was not sonsidered suitable for inclusion in an arithmetical work. Though Mahāvira's voluminous *Gaṇita-Sāra-Saṁgraha* has[3] explicit references to factorisation, the book contains no rule for the purpose. Probably, Śrīpati (1039 A.D.) is[4] the first Hindu writer who has formally treated the topic of finding divisors of a number in his *Siddhānta Śekhara*. Nārāyaṇa is the first Hindu writer who has devoted a full chapter to it in his *Gaṇita Kaumudī*.

INTRODUCTORY VERSE

R.1 "Now, (I) state *Bhāgādāna* (i.e., methods of finding factors or divisors of a number), for the pleasure of mathematicians, knowing which (even an) ordinary (person) at once becomes a mathematician."

DIVISORS OF A NUMBER

R.2-4(a) "If the number is even, divide it by 2, repeatedly, until it becomes odd; if 5 is in the unit's place, divide by 5. If the number is neither even nor 5 is in the unit's place, the prime numbers, 3, 7, 11 etc. should be tried (as divisors). One should find out the divisors until it is possible to do so."

These are ordinary methods for finding the divisors of a number when the divisors are 2, 5, and 3 etc. These rules were earlier given by Śrīpati[5].

DIVISOR IN CASE OF A SQUARE

R.4b. "In case the dividend is a perfect square, its square-root is a twice repeated factor."

The rule follows from the definition of a square. The rule was earlier given by Śrīpati.[6]

DIVISOR IN CASE OF A NON-SQUARE NUMBER

R.5-6(a) "If the dividend is a non-square number, find an optional square number which when added to the dividend, (the sum) becomes a perfect square. The sum and the difference of their square-roots are the divisors of the dividend. The intelligent should now proceed, as before, to find out the divisors of these numbers."

Throughout this chapter, N represents the number whose divisors are to be found out. Let, $N + k^2 = b^2$. According to the rule, $(b + k)$ and $(b - k)$ are the divisors of N. The rule is obvious as then, $N = (b + k)(b - k)$.

Alternative Rule

R.6(b)-7(a) "In case of a non-square number, (if) twice the square-root of its nearest square increased by 1 (and then) lessened by the difference (of the non-square-number and the nearest square-root) be a square, it is to be taken as the number to be added to (the non-square number to) make it a perfect square."

Let, $N = a^2 + r$ . . . .(1) and $2a + 1 - r = b^2$ . . . (2)

Then, according to the rule, $N = (a + b + 1).(a - b + 1)$.

It is so, because, $N + b^2 = (a + 1)^2$, from 91 and (2). The rule was earlier given by Śripati.[7] The rule was[8] rediscovered by the French mathematician Fermat (about 1643 A.D.).

Alternative Rule

R.7(b)-8(a) "In case the above number is not a perfect square, it should be further increased (by the successive terms of the arithmetic series whose) first term is ('the sum of) twice the nearest square-root (of the non - square number) added to 3, (and whose) Common difference is 2, untill itbecomes a perfect square."

The rule is an extension of R.6(b)-7a. With the notations of the above rule [R.6(b)-7(a),] suppose that $2a + 1 - r$ is not a perfect square. Accordng to the rule, in that case, r terms of the series, $(2a + 3), (2a + 5), (2a + 7), \ldots, (2a + 2r + 1)$ be further added to $(2a + 1 - r)$ to make it a perfect square $(= k^2)$.

Rationale : Now we have, $N + k^2 = a^2 + r + 2a + 1 - r + (2a + 3) + (2a + 5) + (2a+7) + .. (2a+2r+1)$

$= a^2 + (2a + 1) + (2a + 3) \ldots (2a + 2r + 1) = (a + r + 1)^2$

So, $N [= (a + r + 1)^2 - k^2] = (a + r + k + 1)(a + r - k + 1)$.

The rule was rediscovered[9] by Fermat (about 1643 a.D.).

Method of Writing Division of a Number

R.8(b)–8(c) "In the case of unequal divisors (having set down the divisors, one after another) multiply the succeeding divisors by the preceding ones and by the product of the preceding ones (taken 2, 3, . . . , all at a time and) set down (the products, thus obtained) ahead of those divisors. In the case of equal divisors (having set down the divisors one after another) multiply each of the succeeding divisors by (the product of) the preceeding ones. Each of these should be severally multiplied by the other divisors, (i.e, by those due to unequal divisors)."

The rule gives a process for writing down all possible divisors of a given umber N.

Let $N = a \times a \times a \times b \times c \times d$.

Then, all the divisors of N are,

(1) b, c, d, bc, bd, cd, bcd.

(2) a, a^2, a^3

(3) $ba, ba^2, ba^3; ca, ca^2, ca^3; da, da^2, da^3; bca, bca^2, bca^3; bda, bda^2, bda^3; cda, cda^2, cda^3; bcda, bcda^2, bcda^3$.

Ex.1 "O mathematician, tell me quickly the numbers by which 2048 is exactly divisble or tell those (numbers) by which 3125 is exactly divisible."

Ex.2 "If you know, tell those (numbers) by which 7520 is exactly divisible (and) those, by which 10201 is exactly divisible."

Ex.3 "O you, proficient in mathematics, if you know the subject of finidng divisors, tell those (numbers) by which 1161 is exactly divisble."

Solution : Here, $1161 = 34^2 + 5$, so, $a = 34$ and $r = 5$. Thus, $(2a + 1 - r =) 2 \times 34 + 1 - 5 = 8^2$ and so, $1161 + 8^2 = 35^2$ or $1161 (= 35^2 - 8^2) = 27 \times 43 = 3 \times 3 \times 3 \times 43$.

Therefore, all possible divisors of 1161 are 3, 9, 27, 43, 129, 387, 1161.

Ex. 4 "If you are fully proficient in the subject of finding divisors, tell quickly, those (numbers) by which 1001 is exactly divisible."

Solution : Here $1001 = 31^2 + 40$, *so*, a = 31 and $r = 40$. Thus, $(2a + 1 - r =)$ $2 \times 31 + 1 - 40 = 23$, is not a square.

Also, $(2a + 3 =) 65$ and $63 - 40 + 65 + 67 + \ldots + 89 = 32^2$

So, $1001 + 32^2 = 45^2$. Thus, $1001 (= 45^2 - 32^2) = 77 \times 13 = 7 \times 11 \times 13$, therefore, all possible divisors of 1001 are, 7, 11, 13, 77, 91, 143 and 1001.

Ex. 5 "O friend, if you are fully proficient in mathematics, tell quickly those (numbers) by which 4620 is exactly divisible."

Ex.6 "O friend, if you are well versed in mathematics, tell me quickly those (numbers) by which 3927 is divisible without a remainder."

Alternative Shorter Method

R.9-10. "The nearest square-root (of the given number) as diminished by an optional number is the divisor. If 'the square of the optional number plus the residue of the square-root' is exactly divisible by the divisor, the given number must be divisible by the same. If not, one should choose another (appropriate) optional number applying one's own intellect."

Explanation and Rationale : The rule states that if, $N = a^2 + r$ and if r is such that $m^2 + r$ is exactly divisible by $a - m$, then N is also exactly divisible by $a - m$. With the above notations, we can write,

$$N = a^2 - m^2 + m^2 + r = (a + m)(a - m) + m^2 + r$$

So, $N/(a - m) = a + m + (m^2 + r)/(a - m)$.

Therefore, if $m^2 + r$ is exactly divisible by $(a - m)$,

N must also be exactly divisible by $(a - m)$.

Ex.7 "O learned, tell quickly, the numbers by which 120 is exactly divisible and also those, by which 231 is exactly divisible."

Solution : Case I. Here, $120 = 10^2 + 20$, so, $a = 10$ and $r = 20$. Choosing $m = 2$, we see that $(a - m =)\,8$ is an exact divisor of $(m^2 + r =)\,2^2 + 20 = 24$. So, 8 must also be an exact divisor of 120. Thus, $120\ (= 8 \times 15) = 2 \times 2 \times 2 \times 3 \times 5$

Therefore, all possible divisors of 120 are

(i) 3, 5, 15; (ii) 2, 4, 8; (iii) 6, 12, 24, 10, 20, 40, 30, 60, 120.

Case II can be similarly dealt with.

RULE FOR TESTING MULTIPLICATION

R.11. "Divide each of the multiplicand and the multiplier by an optional number. Multiply the remainders and (then) divide the product by the optional number. If the remainder so obtained be equal to the remainder obtained on dividing the product (of the multiplicand and the multiplier, then), it is correct."

Let m, m' be the multiplicand and the multiplier. Let r and r' be the remainders, when m and m' are divided by n, severally. Then, according to the rule, if R be the remainder, when mm' is divided by n, and also when rr' is divided by n, then the multiplication is correct.

Rationale : Let,

$$(m/n) = p + (r/n),\ (m'/n) = q + (r'/n),\ (mm'/n) = s + (R/n) \ldots (1)$$

and $(rr'/n) = t + (R'/n) \ldots (2)$

where, (r/n), (r'/n), (R/n) and (R'/n) are proper fractions.

Then, $\left(\frac{mm'}{n^2}\right) = \left(p + \frac{r}{n}\right)\left(q + \frac{r'}{n}\right) = pq + (pr'/n) + (qr/n) + (rr'/n^2)$

So, $(mm'/n) = npq + pr' + qr + t + (R'/n)$, from (2)

or $s + (R/n) = t' + (R'/n)$, from (1), where $t' = npq + pr' + qr + t$.

Therefore, $R = R'$ as s and t' are integers and (R/n) and (R'/n) are proper fractions.

A complete set of rules for checking by nines is first found in India. There[10] is no doubt about the Hindu origin of the check of nines. Nārāyaṇa is probably the first Hindu mathematician to give rules for testing operations by the casting out of any desired number. Avicenna (1020 A.D.) and Maximus Planudes ascribe[11] and Indian origin to the check of nines. Before Āryabhaṭa II, it was probably used to test multiplication and division only. Behā Eddin (1600 A.D.) gives [12] the check of nines in exactly the same form as Āryabhaṭa II.

Ex. 8 "Friend, 29 multiplied by 17 is equal to 493. Tell, (whether this) is correct (or not)?"

Solution : Here, in this case, $m = 29$, $m' = 17$. Let, $n = 3$, then $r = 2$, $r' = 2$. So, $R' = 1$. Also, taking 493 as the product, it being divided by 3 leaves 1 as the remainder which is equal to R' and so the multiplication is correct. Alternatively, let $n = 5$, then $r = 4$, $r' = 2$. So, $R' = 3$. Also, the remainder when 493 is divided by 5 is 3 and so the multiplication is correct. Alternatively, let $\eta = 8$, then $r = 5$, $r' = 1$. So, $R' = 5$. Also, the remainder when 493 is divided by 8, is 5 and so the multiplication is correct and so on.

REFERENCES FOR CHAPTER XI

1-3. Shukla, K.S.: Hindu Methods for finding factors or divisors of a number: *Gaṇita*, Vol 17, No.2, 109.

4-7. *SS*. 134.

8-9. Dickson, L.E., *History of Theory of Numbers*, Vol.I, 357. For further details, refer 3 115-116.

10-12. For further details, see, *HHM*(I), 183-184.

CHAPTER XII

FRACTIONS

In *Bhāga Jāti* of fractions, R.1 is a rule to express 1 or any other number as the sum of n(unit) fractions and R.2 and R.3-4. are the other 2 rules for the same. R.5-6 and R.7 are 2 different generalizations of R.1. A rule to find fractions whose numerators are given and whose sum is 1 has been given in R.8-10(a) an another rule for the same making use of indeterminate anlysis, if necessary has been given in R.10(b-c)-12. A rule to find fractions whose denominators are given and whose sum is 1, making use of indeterminate analysis has been given in R.13-15. In *Prabhāga Jāti* of fractions R.1 is rule to find unknown fractions such that the fractions when added after simplification, the sum becomes 1 and in *Bhāgāanubandha* and *Bhāgāpavāha Jātis* of fractions also R.1 is a rule for the same.

In *Svāṁśānubandha Jāti* of fractions R.1-2(a) is a rule to find fractions of the upper row and R.2(b)-3(a), that of the lower row when the number of rows of the fractions is 2, and R.3(b)-4(a), that of the middle row when the number of rows of the fractions is 3, so that the fractions when added after simplication, the sum becomes 1 in each case. Again, R.4(b)-5(a) treats the case when the number of unknown rows is 2 (or more). R.5(b)-6, is a rule to find the fractions of the upper row and R.7. that of the lower row under above conditions.

The Sanskrit term *bhinna* means broken and stands for a fraction. The European terms *fraction*, *fracto* and *rotto* etc. have[1] been derived from the Latin *fractus* or *ruptus* which means broken. However, the term *bhinna* had[2] a more general connotation and included fractions in combination wuch as

$$\left(\frac{a}{b}\pm\frac{c}{d}\right),\ \left(\frac{a}{b}\text{ of }\frac{c}{d}\right)\ \text{and}\ \left(\frac{a}{b}\pm\frac{c}{d}\text{ of }\frac{e}{f}\right)\ \text{etc.}$$

These forms were temed[3] *Jāti* in Sanskrit which means 'classes'

So far as unit fractions are concerned these were widely used[4] in ancient Egypt. The *Rhind Mathematical Papyrus* (about 1650 B.C.) contains[5] a table in which the fraction $2/n$ has been expressed as a sum of unit fraction for each value of n from 5 to 101. Such tables were[6] needed for the Egyptian dyadic arithmetical process of multiplication and division through duplation mediation. However, how the table was[7] obtained has not been explained in the Papyrus. Mahāvīra in his *Gaṇita Sāra Saṁgraha* has[8] given several rules for converting any number into a sum of unit fractions. Fibonacci (about 1200 A.D.) is said to have used[9] a device for converting a fraction into a sum of unit fractions.

BHĀGA JĀTI

Terminology : Fractions of the form $\left(\frac{a}{b}\pm\frac{c}{d}\pm\frac{e}{f}\pm\ldots\ldots\right)$ were written as $\begin{array}{|c|c|c|}a&c&e\\ b&d&f\end{array}$ ...or $\begin{array}{|c|c|c|}\dot{a}&\dot{c}&\dot{e}\\ b&d&f\end{array}$ ... by the Hindus, where the dots denote subtraction.

TO EXPRESS ANY NUMBER AS THE SUM OF N UNIT FRACTIONS

R.1 "Multiply the consecutive terms of the arithmetic progression whose first term and common difference are 1 each, two at a time. (The products) are the denominators, the last term (of the series) being the last denominator. (The fractions) when added (yield) 1 (as the sum). For any other desired sum, every fraction is to be multiplied by that number."[10]

According to the rule,

$$1=\frac{1}{1.2}+\frac{1}{2.3}+\ldots\ldots\frac{1}{(n-1)n}+\frac{1}{n}\qquad\ldots.(i)$$

$$\text{and } a=\frac{a}{1.2}+\frac{a}{2.3}+\ldots+\frac{1}{(n-1)n}+\frac{a}{n},\qquad\ldots.(ii)$$

where a is any number. The rule is obvious.

Ex.1 "The numerators of 6 terms are 1 each. Their sum is either 1 or 1/2. Tell here, their numerators."

ALTERNATIVE RULE

R.2 "Terms of the Geometric Progression (Whose) first term is 1 and, common ratio is 3, upto the number of places and the first term and the last term, multiplied by 2, are the denominators. The numerator of the last is multiplied by3. (The fractions) when added become 1."[11]

According to the rule,

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{3^2}+\ldots.+\frac{1}{3^{n-2}}+\frac{3}{2.3^{n-1}}$$

ALTERNATIVE RULE

R.3-4 "Add the denominator of the result to such an optional number that the sum becomes exactly divisible by the numerator of the result. The quotient is the denominator (of the first fraction). Subtract this from the result and repeat the process with the remainder, again and again. Each of the remainders and the last remainder are the denominators and 1 is their numerators. This has been made clear by the elders."[12]

Let the sum be (a/b). Let c be an optional number such that $(b+c)$ is exactly by a. Now, $a/(b+c)$ is the first term and the remainder is $(a/b)-a/(b+c)=ac/b\,(b+c)$.

The process is repeated with $ac/b(b+c)$ in place of (a/b). The sum of the terms and the last remainder is equal to the desired sum (a/b). If $a = 1 = c$, the fractions obtained are unit fractions. If $b = 1$ as well, the sum = 1. Moreover, optional number c may be the same, or different for different terms.

Ex.2 "The sum is 5/6 or 7/9. What are their denominators if their numerators are 1 each and their number is 4."

ALTERNATIVE RULE FOR EXPRESSING UNITY AS THE SUM OF N UNIT FRACTIONS.

R.5-6 "Desired numbers are supposed. The first is called *Kand* and the last, *Agra*. Each term multiplied by its previous term (and also) lessened by it, (seprately) happen to be the denominator and the numerator, (respectively), The *agra* happens to be the dnominator and 1, the numerator of the last (term. After that) all the numerators are multiplied by the *kanda*. The sum of the fractions (so obtained) is 1."

Let the desired numbers be, $a_1, a_2, a_3, \ldots, a_n$. According to the rule,

$$1 = a_1 \cdot \left[\frac{a_2 - a_1}{a_2 \cdot a_1} + \frac{a_3 - a_2}{a_3 \cdot a_2} + \frac{a_4 - a_3}{a_4 \cdot a_3} - \ldots + \frac{a_n - a_{n-1}}{a_n \cdot a_{n-1}} + \frac{1}{a_n} \right]$$

The rule is obvious as the sum of the fractions within the bracket $1/a_1$. The rule is a generalisation of R.1.

Ex.3 "O friend, if you know, tell me quickly, 6 fractions, (whose) sum is 1."

To Express 1 as the Sum of n Fraction with Given Numerators

R.7 "Suppose 1 to be the first (term) and each numerator, added to (the sum of all) previous numerators and to 1 to be the other terms. The product of two consecutive terms are the denominators, 1 being the numerator and the last term, the denominator of the last (fraction)."

Let the given numerators be $a_1, a_2, a_3, \ldots a_{n-1}, 1$. According to the rule,

$$1 = \frac{a_1}{(a_1 + 1)} + \frac{a_2}{(a_1 + 1)\,(a_2 + a_1 + 1)} + \frac{a_3}{(a_2 + a_1 + 1)\,(a_3 + a_2 + a_1 + 1)} + \ldots$$

$$+ \frac{a_{n-1}}{(a_{n-1} + \ldots + a_1 + 1)\,(a_{n-2} + \ldots + a_1 + 1)} + \frac{1}{(a_{n-1} + \ldots + a_1 + 1)}.$$

The rule is obvious. If $a_1 = a_2 = \ldots = a_{n-1} = 1$, n unit fractions are obtained whose sum is unity. The rule is a generalisation of R.1.

Ex.4 "Numerators in 4 places are 3 etc. with a common difference of 2 (and the last numerator is 1). Tell the denominators of these fractions, the sum of these fractions being 1."

To Express 1 as the Sum of Fractions whose Numerators are Given

R.8-10(a) "Form (unit) fractions, such that their number is equal to the number of pairs (of the numerators) and their sum is 1. Multiply the denominaotr of such a fraction by the first given numerator (in a pair). (The product) added to the numerator of the other fraction (of the pair) is the denominator of the previous (fraction of the pair) and that (i.e., the sum multiplied by the denominator (of the formed fraction) is the (desired) denominator of the other (fraction of the pair). Repeat (the process, in) all the pairs."[13]

In the case of odd number of fractions, the numerator and the denominator of the last (formed) fraction multiplid by (the numerator of the last given) fraction are the numerator and the denominator of the (desired) last fraction. The numerators of the pairs divided by their own respective denominators (are the desired fraction)."

First of all 1 is separated into m or (m + 1) unit fractions, according as the desired number of terms, whose denominators are required, is even (=2m) or odd

(= 2m +1). After that, each unit fraction is separated into a pair of fractions with given numerators, say, b and c, making use of the formula,

$$\frac{a}{(ab + c)} + \frac{c}{a(ab + c)} = \frac{1}{a},$$

where $1/a$ is the unit fraction. If the number of terms be odd, the numerator and the denominator of the last unit fraction are (each) multiplied by the last (given) numerator. Other fractions are formed as in the case of even number of fractions as indicated above. Obviously, the denominators of the fractions will depend upon the unit fractions which are formed, their sum being 1. The rule is easy to establish. Mahāvira has [14] given the rule for an even number of terms.

Ex. 5 "Tell, what are the denominators of 6 or 7 fractions whose first numerator is 3 and others are 2 more in succession, their sum being 1."

To Express 1 as the Sum of 2 Fractions with Given Numerators

R.10(b-c) "Multiply the first given numerator by the denominator of the result. Add (the product to) the other (given) numerator. Divide the sum by the numerator of the result. (The quotient is) the denominator (of the first fraction and the same, i.e., the quotient) multiplied by the denominator of the result is (the denominator of) the last (fraction)."

R.11-12. "If the division is not without remainder, (supposing) the lesser numerator as the dividend, the other (numerator) as the additive and the numerator of the result as the divisor, the quotient with its additive, obtained by the method of indeterminate analysis will be the denominator. The denominator of the result is divided (or) multiplied by the denominator (obtained by the indeterminate analysis) according as it is lesser or greater than (the denominator of the result. If this division is not without remainder, the solution is not possible."

Let the numerators of the desired fractions be a and b. (In case, unit fractions are desired, ($a = b = 1$). Also, let their sum be m/n. According to R.10(b-c), $\left(\frac{an + b}{m}\right)$ and $n\left(\frac{an + b}{m}\right)$ are the denominators of a and b, respectively. This is so, as

$$\frac{a}{\left(\frac{an + b}{m}\right)} + \frac{b}{n\left(\frac{an + b}{m}\right)} = \frac{m}{n}$$

Mahāvira, besides this rule[15]. has also given an alternative rule[16] for the same. According to R.11-12, if $(an + b)/m$ is not integral, it has to be made integral by solving $(an + b)/m = y$, an integer. The rule has been made clear by the solution of example 8, given later on.

Ex. 6 "If you know the miracles of denominators, tell quickly, the denomiators of (two fractions), whose numerators are 1 each and (whose) sum is 1/20.

Ex. 7 "If you know mathematics, tell the denominators of two fractions, whose numerators are 3 (and) 7 and whose sum is 2/5."

Ex.8 "Friend, tell quickly the denominators of two fractions, whose numerators are 3 and 5 and whose sum is 71/70."

Here, the denominator of the first fraction, by R.10(*b-c*) is 215/71 which is not integral. So, now R.11-12 is to be applied and therefore, the equation $(3n + 5)/71 = y$, is to be solved. Its solution will be, $n = 22 + 71t$ and $y = 1 + 3t$, where t is any integer. Now, if we take $t = 3$, the quotient $(= y)$ becomes 10 and $(3n + 5)/71$ becomes integral. So, 10 will be the denominator of the first fraction. Also, since, 10 < 70 (= the denominator of the result), so (70/10 =) 7 will be the other denominator. Thus, the fractions will be 3/10 and 5/7 According to Nārāyaṇa, "At times the denominators are obtained by supposing the additive to be negative."

To Find Numerators When Denominators Are Given

R.13-15. In the case of unknown numerators, 1 is, separately, supposed for (each) of the numerators. (Then, the fractions and their sum) are reduced to a common denominator and the common denominator is removed. Now, all the numerators are supposed except any two of them. Any one out of (the remaining) two is the divisor, the negative of the other, the dividend and 'the numerator of the (given) sum less (the sum of) the supposed numerators' the additive. After that, reducing the indeterminate equation (i.e., reducing the dividend, the divisor and the additive by their common measure, if any) the multiplier and the quotient along with their additives are obtained. (These along with the supposed numerators) are the numerators of the (given) denominators."

Explanation : Let, $b_1, b_2, \ldots, b_n$ be the given denominators and a/b, the sum of the fractions. First of all, suppose 1 to be the numerator of each of the fractions. Add all the fractions, so that,

$$\frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_n} = \frac{r_1}{s_1} \text{ (say).}$$

Also, reduce these to a common denominator. Suppose, we get,

$$\frac{a_1}{d} + \frac{a_2}{d} + \ldots\ldots + \frac{a_{k-1}}{d} + \frac{a_k}{d} + \ldots + \frac{a_n}{d} = \frac{r_2}{d} \qquad (1)$$

Let d be removed everywhere and, all the numerators (except any two, say at the places occupied by a_{k-1}, a_k) be $m_1, m_2, \ldots, m_n$ and those at a_{k-1}th and a_kth places be x and y, respectively. Also,

$$r_2 - [m_1a_1 + m_2a_2 + \ldots + m_{k-2}a_{k-2}a_{k+1}a_{k+1} + \ldots + m_na_n] = s \text{ (say)}, \qquad \ldots (2)$$

Now, solve the equation, $(-a_{k-1}x + s)/a_k = y$ $\ldots (3)$

Suppose, its solution is, $x = h + a_kt$, and $y = c - a_{k-1}t$ $\ldots (4)$,

where t is any integer. Then,

$m_1, m_2, \ldots, m_{k-2}, h + a_kt, c - a_{k-1}t, m_{k+1}, \ldots, m_n$ will be the desired numerators, in order.

Rationale : Here, $a_{k-1}(h + a_kt) + a_k(c - a_{k-1}t) = a_{k-1}x + a_ky = s$, from (4) and (3)

So, $m_1a_1 + m_2a_2 + \ldots + m_{k-2}a_{k-2} + s + m_{k+1}a_{k+1}x + \ldots m_na_n = r_2$, from (2). Now, from (1), the right side is the sum of the fractions, when these are reduced to a common denominator and the denominators are removed and hence the rule.

Ex.9 The numerators (of the fractions) are destroyed, whose denominators are 5,8,9 and 12 and whose sum is 3/(1/40). Tell quickly their numerators in many ways, (for you are like) a lion (i.e., a highly praised mathematician) always capable of roaring loudly to shake a praiseworthy (but) madly intoxicated elephant (i.e., a bad mathematician)."

Solution : Here, the denominators are 5, 8, 9 & 12 and the sum is 3+1/40. Supposing 1 to be the numerator of each of the fractions, these become 1/5, 1/8, 1/9 and 1/12. Reducing these and thesum of a common denominaotr, we get, 72/360, 45/360, 40/360, 30/360 and 1089/360 and after removing the denominators, these become, 72, 45, 40, 30 and the sum, 1089.

Now, let the first two numerators be supoosed as 2 and 1. Then, we have, $1089-(2\times72+ 1\times45) = 90$ So, we have to solve the equation, $(-40x+ 900)/30 = y$ or, $-4x + 90 = 3y$.

Its solution will be, $x = 0 + 3t$ and $y = 30 - 4t$, where t is any integer, i.e., $x = 0, 3, 6, \ldots$ and $y = 30, 26, 22, \ldots$, in order. So, the desired numerators will be, 2, 1, 0, 30 or 2, 1, 3, 26 or 2, 1, 6, 22 etc. If the two numerators are supposed differently, other results will be obtained. Again, if other numerators are supposed, then again other results will be obtained and if differently different results will come and so on.

PRABHĀGA JĀTI

To Find Unknown Fractions

Terminology : Fractions of the form $\left(\frac{a}{b} \text{ of } \frac{c}{d} \text{ of } \frac{e}{f} \ldots\right)$ were written as $\begin{array}{|c|c|c|}a & c & e \\ b & d & f\end{array}$ .. by the Hindus and were named *Prabhāga Jāti*.

R.1 "Divide the fractions whose sum is the desired result by the given fractions, (separately). The quotients) happen to be the (desired) fractions. In this way, in the case of many terms, multiply the product of the optional terms by thegiven terms (and then) divide."

Suppose,

$\frac{1}{a}+\frac{1}{b}+\ldots.=1$ ...(1) and also, that (c/d), (e/f), $g/h)$, ... be the given fractions, the number of such fractions being the same as that of those given in (1). Let (i/j), (m/n), ..., be the desired fractions. Now, if we have, $(c/d).(i/j)=\frac{1}{a}$, then, $(i/j)=(1/a).(d/c)$ Thus, (i/j) can now be determined, since the right side is known. Similarly, (m/n) etc. can be determined. The general case, i.e., to express $1/a$ as $(c/d).(i/j).(k/l)$... can be similarly dealt with and hence the rule.

Ex.10 "If you know the *Prabhāga(Jāti* of fractions), tell quickly the fractions, 3/4 of which (i.e., of the first), 4/5 of which (i.e., of the second) and 6/5 of which (i.e., of the third), when added, become1"

Ex.11 "A rich man gave a person, 6/7 of a fraction of a fraction of a fraction, 2/9 of a fraction of a fraction of a fraction and 3/10 of a fraction of a fraction of a fraction. The Brahman get 1. Friend, tell the fractions."

In this example certain other optional fractions will also have to be taken. So, the result will depend on both the type of optional fractions taken, i.e., the fractions whose sum is 1 and the fractions whose products with the given fractions equal the former optional fractions.

BHĀGĀNUBANDHA JĀTI AND BHĀGĀPAVĀHA JĀTI

Terminology : Fractions of the forms, $a+\frac{b}{c}$ and $a-\frac{b}{c}$ were written as, $\begin{array}{|c|}\hline a\\\hline b\\\hline c\\\hline\end{array}$ and $\begin{array}{|c|}\hline a\\\hline b\\\hline c\\\hline\end{array}$ by the Hindus and were named *Bhāgānubandha Jāti* and *Bāgāpavāha* Jāti, respectively. If there were more than one column of such fractions, fractions in each of the columns were simplified, separately, and then added or subtracted, as the case might be.

To Find Unknown Fractions

R.1 "Place optional integral numbers, separately. Subtract their sum from the result. Put positive or negative fractions whose sum is equal to the remainder, below them."

Rationale : Let, $a\frac{e}{f}+b\frac{g}{h}+c\frac{i}{j}+d\frac{k}{l}+\ldots\ldots=s$ ....(1)

Also, suppose that, $a, b, c, d, \ldots$ .are optional numbers. Then, from (1),

$e/f+g/h+i/j+k/l+\ldots\ldots=s-(a+b+c+d+\ldots)$ ....(2)

Now, right side of (2) being known, (e/f), (g/h), ... can be obtained by giving them values, whose sum is equal to the right side of (2), The other case can be similarly dealth with by replacing '+' by '–' at proper places.

Ex. 1 "Fractions at 4 places, added to or subtracted from (integral) number become 12. Tell them."

SVĀMŚĀNUBANDHA *JĀTI* AND SVĀMŚĀPAVĀHA *JĀTI*

Terminology : Fractions of the forms,

$$\frac{p}{q}+\frac{r}{s}\text{ of }\frac{p}{q}+\frac{t}{u}\text{ of }\left(\frac{p}{q}+\frac{r}{s}\text{ of }\frac{p}{q}\right)+\ldots\ldots\text{ and}$$

$$\frac{p}{q}-\frac{r}{s}\text{ of }\frac{p}{q}-\frac{t}{u}\text{ of }\left(\frac{p}{q}-\frac{r}{s}\text{ of }\frac{p}{q}\right)-\ldots\ldots\text{ and}$$

written as $\begin{array}{|c|}\hline p\\ q\\\hline r\\ s\\\hline t\\ u\\\hline\end{array}$ and $\begin{array}{|c|}\hline p\\ q\\\hline r\\ s\\\hline t\\ u\\\hline\end{array}$ by the Hindus and were named *Svāṁśānubandha Jāti* and *Svāṁśāpavāha Jāha Jāti*, respectively.

Here also, if there were more than one column of such fractions, fractions in each of these columns were simplified, separately, and then added.

Rationale : 1 is separated into as many fractions as there are columns in the given fractions. x is supposed in the unknown place in the first column of the given fraction and the fractions of the first column are simplified (by the method of *svāṁśānubandha* or *svāṁspavāha*, as the case may be) and then equated with the first fraction into which 1 has been separated. This forms a linear equation from which the value of x is obtained. Similarly values of all the other unknown fractions are obtained. Nārāyaṇa does the same by supposing 1 in the place of x. As fraction of each column equals each of the frations into which 1 has been separated, in order, so, the sum of the fractions becomes equal to1.

To Find Fractions of Upper Row

R.1-2(a) "If the fractions are lying below, keeping 1, above them, separately, these are simplified by the method of *svāṁśānubandha*. The fractions, whose sum is 1, divided by the simplified fractions, happen to be the fractions standing above those fractions.[17]"

Ex. 1 "1/2, 1/4, 1/6, 1/7 (and) 1/8 respective parts of certain fractions along with them, in order, when added, become 1. O elder, if you know, tell me the fractions in the upper part (in *svāṁśānubandha* type of fractions)."

To Find Fractions of Lower Row

R.2(b)-3(a) "Divide the fractions whose sum is 1 by the fractions standing above, separately. Subtract 1 from (each of) them (the remainders) happens to be the fractions lying below."[18]

Ex.2 "1/5, 1/8, 1/16 (and) 1/9 (are added) to certain fractions of themselves, in fractions, tell quickly, the fractions lying below."

To Find Fractions of Middle Row

R.3(b)-4(a) "Subtract the numbers standing above from the numbers obtained by the method stated earlier for fractions. Then, again divide (the remainders) by them (i.e., by the numbers standing above). (The quotients) are the fractions lying in the middle."

Here, "By the method stated earlier for fractions" means fractions obtained by R.1-2(a), i.e., keep 1 in the unknown places in the unknown middle row. Then simplify the fractions of the middle row and the last row by the method meant for *svāṁśānubhandha* type of fractions. Finally divide the fractions whose sum is 1 by those simplified fractions, in order. Nārāyaṇa gives an alternative method for finding fractions lying in the middle. According to him, divide the fractions obtained by R.1-2(a), by the fractions of upper row and then subtract 1 from each of the quotients, one by one and in order. The remainders are the fractions lying in the middle.

Ex.3 "1/5, 1/8 (and) 1/6 added to certain fractions of those, (in order), are again added to 1/6, 1/4 and 1/2 of the resulting fractions, (in order). (The results) when added, become 1. Tel them (i.e., the fractions) to me, quickly."

To Find Fractions of Two (or More) Rows

R.4(b)-5(a) "Keep optional fractions in places lying above, where these are not known. Obtain (the fractions) lying below, where (these) are not known, by the method stated earlier."[19]

According to the rule, if fractions in more than 2 rows are not known, fractions in 1 row (preferably the upper row) should be supposed in an arbitrary manner and then the method stated earlier should be followed. Obviously, this method can also be followed if fraction in more than 2 rows are unknown. Here, Nārāyaṇa, comments, "Fractions are produced in many ways, according as optional numbers (i.e., fractions) are supposed."

Ex.4. "1/3, 1/2 (and) 1/4 of what fractions, added to these and the resulting fractions added to 1/4, 1/5, and 1/6 of the resulting fractions; (the final resulting) fractions of what (other) fractions, added to (the final resulting) fractions (in order), when added become 49."

To Find Fractions of Upper Row

R.5b-6 "If fractions lying below are given, place 1 above them, separately. (After) simplifying (them) by the method of *svāṁśapavāha*, divide them by the fractions whose sum is 1, (separately and in order. The quotients) happen to be the fractions lying above."[20]

Ex.5 "1/8, 1/7, 1/6, 1/4 (and) 1/2 of what fractions, subtracted from them, in order, when added become 1. O mathematician, if you have pride in mathematics and know, here tell them."

To Find Fractions of Lower Row

R.7 "Divide the fractions whose sum is 1, separately, by the fractions standing above. (in order). Subtract (the quotients), separately, from 1. The remainder are the fractions lying below."[21]

Ex.6 "If you are an expert in fractions, tell, 1/2, 1/5, 1/4 (and) 1/3 of what fractions, subtracted from them, when added, become 1."

Example Based on Earlier Rule

Ex.7 "Certain fractions of 101/2, 101/3 (and) 101/4 are subtracted from these, in order and 1/2, 1/4 (and) 1/5 of the resulting fractions are subtracted from the resulting fractions, in order. O elderly learned, if you are confident, tell quickly, what fractions of the final resulting fractions, when added, become 1/2."

The example is based on R.4-5(a).

References For Chapter XII

1-3. *HHM*(I), 188

4-5. Gupta, R.C., The Mahāvīra-Fibonacci Device to Reduce (p/q) to Unit Fractions, *History and Pedagogy of Mathematics Newsletter,* July 1993, No. 29, 10-12.

6. Refer 5 for details and examples.

7. Refer 5, 11.

8. *GSS*, 52-67.

9. Refer 5, 12.

10. See, *GSS*, 49-50.

11. Cf., *GSS*, 52.

12. Cf., *GSS*, 54-55.

13. Cf. *GSS*, 57-58.

14-15. Cf. *GSS*, 57.

16. Cf. *GSS*, 56. See also, *HHM*(I), 202-203.

17. See also, *GSS*, 62-63.

18. See also, *GSS*, 63.

19-21. See also, *GSS*, 65-66.

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THE GAṆITA KAUMUDĪ OF NĀRĀYAṆA PAṆḌITA
CHAPTER - XIII
ENGLISH TRANSLATION WITH NOTES

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TRANSLATOR'S INTRODUCTION[1]

Nārāyaṇa Paṇḍita wrote his *Gaṇita Kaumudī* in 1356 A.D. Thirteenth Chapter of it has been named *Aṅka Pāśa* by its author. *Aṅka Pāśa* means concatenation (net or chain) of numbers. The chapter deals with combinatorics, i.e. permutations, combinations, partitions, sequences of bionomial and polynomial coefficients, binomial theorem, figures of numbers containing these sequences and relations between them.

In India earliest example of a problem on combinatorics and its solution is[2] found in medical treatises *Caraka Saṁhitā* and *Suśruta Saṁhitā*. However, here, systematic study of combinatorics originated as a hand maid of Sanskrit (and Prakrit) prosody and these concepts have been developed and utilised by the authorities on metrical sciences.

The word *mātrā* in Sanskrit signifies measure. The basic units in Sanskrit prosody are letters having a single *mātrā* (mora) and those having two *mātrās* (morae) called *Laghu* (light) and *Guru* (heavy), respectively. The former is represented by | and the latter by ʃ resembling to a great extent to capital Roman letters I and S, respectively and so for convenience, throughout this paper, these have been represented by these capital letters. Their role in prosody is the same as that of 1 and 2, in combinatorics. Thus, originated the mathematics of combinatorics in India.

Usually, the last chapter of books on Sanskrit prosody contains a mathematical treatment of Sanskrit metres. In this connection, it is very interesting to note that besides the *Gaṇita Kaumudī* and Bījagaṇitāvataṁśa, Nārāyaṇa Paṇḍita also wrote a book on Sanskrit prosody, viz. *Maṇi Mañjarī*.[3]

Though there is[4] a long list of authors of Sanskrit prosody prior to Ācārya Piṅgala (c. 200 B.C. or earlier) his *Chandaḥ Śāstra (Sūtra)* is the earliest extant

work on the subject. Ācārya Bharata's *Nāṭya Śāstra* (between 100 B.C. – 350 A.D.) is the next available work on the subject. The other available works on the subject are the *Agni Purāṇa* (4th century A.D. or earlier), *Janāśrayī Chandoviciti* of Janāśraya (prior to 600 A.D.), the *Viṣṇudharmottara Purāṇa* (7th century A.D.), *Vṛttajātisamuccaya* of Virahāṅka (between 600 – 800 A.D.), *Jayadeva Chandas* of Jayadeva (between 600 A.D. – 900 A.D.), the *Ratna Mañjūṣā* (c. 800 A.D.) *Chandonuśāsana* of Jayakīrti (c. 1000 A.D.), *Vṛtta Ratnākara* of Kedāra Bhaṭṭa (c. 1075 A.D.), *Chandonuśāsana* of Hemacandra (1150 A.D.), and *Prākṛta Paiṅgala* (c. 1310).

In mathematics Brahmagupta (628 A.D.) has[5] devoted a full chapter to the topic. Mahāvīra (850 A.D.) has[6] also dealt with the topic. Nārāyaṇa has[7] also devoted a full chapter to combinatorics.

The following abbreviations are used.

Ag.	*Agni Purāṇa* of Maharṣi Vedavyāsa, edited by B. Upadhyaya, Chowkhamba Sanskrit Series Office, Varanasi, 1966.
Bh.	*Nāṭya Śastra* of Bharata Muni with the commentary of Abhinavagupta, edited by M. Sastri, Part II, Benares Hindu University, Varanasi, 1975.
BSS.	*Brāhma Sphuṭa Sidhānta and Dhyānagrahopadeśādhyāya* by Brahmagupta, edited with his own commentary by Sudhakara Dvivedi, Benares, 1902.
$C(q, r)$	Number of combinations of q different digits (or things) taking r of them at a time.
Ex.	Example (*Udhāraṇa*).
GB.	*Gaṇita Bhāratī.*
GK.	*Gaṇita Kaumudī* of Nārāyaṇa Paṇḍita edited by Padmakara Dvivedi, Benares, Part I, 1935, Part II, 1942.
GSS.	*Gaṇita Sāra Saṁgraha* of Mahāvīra edited with Hindi translation by L.C. Jain, Sholapur, 1963.
He.	See Ja.
HHM.	*History of Hindu Mathematics* by B.B. Datta and A.N. Singh, Lahore, Part I, 1935, Part II, 1938.
Ja.	*Jayadāman* edited by H.D. Velankar, Haritosha Samiti, Bombay, 1949 and containing,
Jd.	(i) *Jayadevachandas* of Jayadeva;
Jk.	(ii) *Chandonusaśāna* of Jayakīrti;

VR. (iii) *Vṛttaratnākara* of Kedāra; and

He. (iv) *Chandonuśāsana* of Hemacandra.

Jn. *Janāśrayi Chandoviciti*, Published by the Curator, The University Manuscript Library, Trivandrum, 1949.

L. *Līlāvatī* of Bhāskara II, translated by H.T. Colebrooke, Kitab Mahal, Allahabad, 1967.

MS. *Mahāsiddhānta* of Āryabhaṭa II, edited by S. Dvivedi, Benares, 1910.

n. The number to be partitioned or
Sum of digits of a particular permutation or partition until otherwise stated.

N P. *Nārada Purāṇa* edited by Sri Ram Sharma, Sanskriti Sansthan, Barailly, 243003, 1981.

p. Number of places (i.e. number of digits in a particular permutation (or partition).

PG. *Pāṭīgaṇita* of Śrīdhara, edited with an ancient commentary and translation by K.S. Shukla, Lucknow, 1959.

P(H). *Chandaḥśāstram* with the commentary of Halāyudha, Parimal Publications, Delhi, 1981.

pp. Pages

P(q, r). Number of permutations of q different digits (or things) taking r of them at a time.

Pr. Pa. *Prākṛta Paiṅgala* edited by B.S. Vyas, Prakrit Text Society, Varanasi-5, 1959.

P(Y). *Chandahsūtrabhāṣyam Yādavaprakāśakṛtam* edited by Haridas Sinharay, The Asiatic Society, Calcutta, 1977.

q. Greatest Digit. All the digits from 1 upto q take part in permutations (or partitions).

Rm. *Ratnamañjūṣā* with a commentary edited by H.D. Velankar, Bhartiya Jnanapitha, Varanasi, 1949.

R. Rule (*Sūtra*).

$u(p, q, r)$. rth term of the *sūcī (nārācikā)* sequence, i.e., coefficient of x^r in the expansion of the polynomial $(1 + x + x^2 + \ldots\ldots + x^{q-1})^p$. $r = 0, 1, 2, \ldots\ldots, p(q-1)$;

VJS. *Vṛttajātisamuccaya* of Virahāṅka with the commentary of Gopāla edited by H.D. Velankar, Rajasthan Oriental Research Institute, Jodhpur, 1962.

VP. *Viṣṇudharmottara Purāṇa*, edited by Priyabala Shah, Vol. I, Part III, Chap. III, Oriental Institute, Baroda, 1958.

VR. See Ja.

Nārāyaṇa first introduces the subject (R. 1) and then states its applications (R. 2). After that he states the names of different sequences [R. 3–5a(i)], those of different figures of numbers [R. 5a(ii)–5b(i)], and those of different processes [R. 5b(ii)]. The different results which are to be obtained [R. 6–9a(i)] and their different varieties [R. 9a(ii)–9b] are the next topics of treatment.

Next he defines and explains arithmetic progression [R. 10(a)], *vyantara paṅkti* (R. 10(b), *vaiśleṣiṇi* sequence [R. 11a], *sārpiṇī* sequence (R. 11b), *jalaukā* sequence (R. 12), *sāmāsikā* sequence [R. 13–15(a)], *pātāla* sequence [R. 15b–17a], geometric progression [R. 17b–18a], *abhyāsikī* sequence [R. 18b–21a], sequence of polynomial coefficients [R. 21b–22a], *yaugika* sequence [R. 22b–23a], and partial sequence of polynomial coefficients [R. 23b–25a]. After that he describes figures of numbers, which have been named *Khaṇḍa Meru* (R. 25b–28), *Patākā* [R. 29–36a], *Sumeru* (R. 36b–39) and *Matsya Meru* (R. 40–43) by him. In R. 44a–44b(i) he gives another method for the formation of sequence of polynomial coefficients and in R. 44b(ii)–44c(i) he gives a property of the sequence of coefficients while in R. 44c(ii)–45a, a relation between polynomial coefficients and generalized socalled Fibonacci numbers.

In R. 45b–47a, the author deals with permutations of *n* different digits (or things) taking all at a time in which R. 45 gives a rule for finding the total number of permutations, R. 46, that for obtaining the number of permutations ending in a particular digit and R. 47(a), that for finding the sum of digits occupying a particular place in all permutations. In R. 47b–48a, he gives rules for the above type of permutations along with permutations of *n* digits taking all at a time if the digits are not all different, in which R. 47b gives a rule for finding the sum of all permutations and R. 48a, that for obtaining total number of digits in all permutations while R. 48b is meant for latter type of permutations.

In R. 49–51a, Nārāyaṇa Paṇḍita gives a rule for the serial arrangement of all possible permutations of *n* different digits taking all at a time, in R. 51b–53, that for finding the unknown serial number, if the corresponding permutation is known and in R. 54–55a, that for finding the unknown permutation, if its serial number is given in such an arrangement. In R. 55b–57, the author again treats the topic of permutations of *n* digits taking all at a time when the digits are not all different, in which R. 55b gives a rule for finding total number of permutations, R. 56a, that for finding the number of permutations ending in a particular digit, R. 56b, that for finding the sum of the digits occupying a particular place in all permutations and R. 57, that for the serial arrangement of all permutations.

In R. 58, he gives a rule to find the values of $C(n, r)$, the number of combinations of n different things taking r at a time and in R. 59a he gives an alternative rule to find coefficients. In R. 59b–61, he once again deals with permutations of n digits taking all at a time when the digits are not all different, in which in R. 59b–60 he gives a rule for finding the unknown permutation if its serial number is known and in R. 61, that for finding the unknown serial number, if the corresponding serial number is given, in the serial arrangement of all permutations with the given digits.

In R. 62–67a he deals with permutations of n digits taking r at time when a digit can be repeated any number ($\leq r$) of times in which, in R. 62–63a, he gives rules for finding the number of permutations along with the number of permutations ending in a particular digit and also the sum of digits at a particular place in all permutations etc., in R. 63b–64, that for serial arrangement of all permutations, in R. 65–66, that for finding the number of permutations, without 1, with 1 once, twice etc. and in R. 67a, an alternative rule for the same with the help of *meru*.

In R. 67b, Nārāyaṇa describes a form of 'the Binomial Theorem', inversely. In R. 68–74, he again deals with the type of permutations described in R. 62–67a, in which R. 68 gives a rule for finding the number of permutations having a given sum, R. 69, that for getting the number of permutations having a given sum and that ending in a particular digit, R. 70, that for obtaining the serial number of permutations having a given sum. Again in R. 71–72a, he gives a rule for obtaining the unknown permutation, if its serial number is given and in R. 72b–74, that for finding the serial number, if the corresponding permutation is given, in the serial arrangement of all permutations under the given constraints.

In R. 75–80, Nārāyaṇa Paṇḍita deals with partitions of a number in which in R. 75a, he establishes a relation between total number of partitions of n and a term of the sāmāsikā sequence, in R. 75b, a relation between number of partitions of n ending in a particular digit and term of sāmāsikā sequence, in R. 76, that between the number of a particular digit in all partitions of n and the term of a pātāla sequence and in R. 77–78, that between the number of p–partition of n and the term of the sequence of polynomial coefficients. In R. 79–80, he states a rule for the serial arrangement of all partitions of n.

In R. 81–83, he gives a rule for the formation of a figure of numbers named *Unmeru* by him. In R. 84–86a, the author while dealing with partitions of a number n, gives a rule for obtaining the unknown partition, if its serial number is given and in R. 86b, that for finding the serial number, if the corresponding partition is given, in the serial arrangement of all partitions of n under the given constraints.

Again in R. 87, while dealing with permutations of *n* different digits taking *r* at a time he gives a rule for their number, another for their sum and still another for the sum of all the digits occupying a particular place. In R. 88–92, the author deals with permutations in fundamental (ascending) order (i.e., combinations, so to say) of *n* different digits taking *r* at a time in which [in R. 88a–b] he gives a rule to find their number, in R. 88c–d – 89a–b that for their serial arrangement, in R. 89c–d– 90, that for movement of shots, in R. 91, that for obtaining the combination if its serial number is given and in R. 92, that for finding the serial number if the corresponding combination is known in the serial arrangement of all such combinations. In R. 93, Nārāyaṇa Paṇḍita states the limitations of the rules and R. 94–97 are his concluding remarks on the chapter.

GAṆITA KAUMUDĪ CHAPTER XIII

INTRODUCTION BY THE AUTHOR

R. 1–9 "For the pleasure of mathematicians, (I) now describe briefly (the subject of) *aṅka-pāśa* (concatenation of numbers, i.e., combinatorics) where bad, wicked and intoxicated (with false sense of vanity) mathematician's vanity shatters. The knowledge of *aṅka-pāśa* is very useful in dramatics, prosody, medicine, garland-making, architecture and mathematics. (In this subject) there are sequences like *caya* sequence (arithmetic progression), *vyantara paṅktī*, *vaiśleṣiṇī* sequence (sequence of units), *sārpaṇikā* sequence (sequence of augmented units), *jalaukikā* sequence, *sāmāsikā* sequence, *pātāla* sequence, *guṇakottara* sequence (geometric progression), *abhyāsikā* sequence, *sūcī* (or *nārācikā*) sequence (sequence of polynomial coefficients), *khaṇḍa sūcī* (or *nārācikā*) sequence (partial sequence of polynomial coefficients), *yaugika* sequence and (figures of numbers like) *khaṇḍameru*, *patākā*, (*su*)*meru*, *matsya-meru* and processes like the movements of shots etc.

Then, there are *pratyayas* (conceptions) of number (of permutations etc.), repetition (of digits), the sum of digits at a particular place added vertically, total sum, total number of digits, (and serial) arrangement (of permutations) and also considerations of number of places (and) of places (and) calculations of number of permutations ending in a particular digit etc., the common difference and the varieties of sums (in this subject).

(Again, here) there are calculations of (numbers of permutations etc.) without 1, having 1, once, twice, thrice etc. and those of the same ending in 1, 2, 3 (and) 4 etc. These and others are *pratyayas* (and again) each of these *pratyayas* are of many varieties. Rules related to them have been stated more clearly as follows."

According to Nārāyaṇa, the knowledge of combinatorics has its use in prosody, dramatics and medicine etc. The word *paṅkti* has been used by Nārāyaṇa

mostly for a sequence besides that for a number in the case of *vyantara paṅkti*. These sequences along with the figures of numbers and the process (i.e., the movement of shots) stated in the rules given above will be explained, later on, at proper places.

SEQUENCES

Arithmetic Progression, *Vyantara Paṅkti, Vaiśleṣiṇī* Sequence (Sequence of Units), *Sārpiṇī* Sequence (Sequence of Augmented Units) and *Jalaukikā* Sequence.

R. 10–12 "(The succession of) numbers with 1 as the first term (and) 1 as the common difference upto the number of places is called *Pracaya* (or *Caya*) Sequence (i.e., Arithmetic Progression). Now, that number (i.e., 1 written) without separation happens to be the *vyantara paṅkti*. That (i.e., 1) also, if written separately, should be known as the *vaiśleṣiṇī* sequence (i.e., the sequence of units). That sequence, at 1 more place (i.e., 1 more term) happens to be the *sārpiṇī* sequence (sequence of augmented units). Leaving the last number of the sequence of augmented units, as it is, the sum of as many terms of the sequence as there are identical digits in the number is called *Jalaukikā* Sequence."

Here all the sequences have been defined for their use in dealing with permutations etc. Consider the permutations of n digits say, $d_1, d_2, d_3 \ldots\ldots, d_n$, taking p of them at a time. Then the number of places is p. All the above sequences have been defined and explained for such p places. The arithmetic progression for p places will be $1, 2, 3, \ldots\ldots, p$.

Vyantara Paṅkti is a number and not a sequence and for p places it is $111 \ldots\ldots 1$, the number of unities being p, i.e.,

$$10^{p-1} + 10^{p-2} + \ldots\ldots + 10^2 + 10 + 1.$$

Vaiśleṣiṇī sequence is $1, 1, 1, \ldots\ldots, 1$ the number of unities being p. Sequence of augmented units is $1, 1, 1, \ldots\ldots, 1$, the number of unities being $p + 1$.

Jalaukikā (*Jalaukā*) sequence is obtained from the sequence of augmented units (for such a place) by arranging the digits in the ascending order and then leaving the last unity as it is, the unities at the places where the digits are equal are added and then these are written in the reverse ordeR. Consider the case when there are 6 places to be filled by the digits $d_1, d_2, \ldots\ldots, d_6$ such that $d_1 < d_2 = d_3 = d_4 < d_5 = d_6$. Then, by doing so as directed above, the *jalaukā* sequence will be 2, 3, 1, 1.

Ex. 1–2 "If you know concatenation of numbers, tell quickly, how the arithmetic progression, the sequence of units, the sequence of augmented units and the *jalaukā* sequence look, the number of places being 4 in all the cases and if the 2 smallest digits are equal."

SĀMĀSIKĪ (OR *SĀMĀSIKĀ*) SEQUENCE

(A GENERALIZATION OF SOCALLED FIBONACCI SEQUNENCE)

R. 13–15a. "Putting unity twice, write their sum ahead of that. Write the sum of numbers, from the reverse order (and in) places equal to the greatest digit, ahead of that. In the absence of numbers in places equal to the greatest digit, write ahead the sum of numbers (in available places). The number of numbers in a *sāmāsikī* sequence happens to be '1 more than the sum'."

As we shall see later on, Nārāyaṇa has used his *sāmāsikā* sequence in partitions of a number n where all the digits from 1 upto q take part in the partitions. He has called q, the Greatest Digit and n, the Sum of Digits. We shall discuss this in detail, later on in R. 75, onwards. Moreover, in permutations of digits in which all the digits from 1 upto q take part and any digit may be repeated any number ($\leq p$) of times, p being the number of places, also, q is called 'the Greatest Digit' (R. 62, onwards).

Let $V(q, r)$ denote the rth term of the sequence, where q is the greatest digit taking part in the partition. Also let the number to be partitioned (i.e., the sum) be n. According to the rule,

$$V(q\,,1) = 1 = \mathrm{V}(q\,,2)$$

$$V(q, r) = V(q, r-1) + V(q, r-2) + \ldots\ldots + V(q, 2) + V(q, 1), \text{ if } 3 \leq r \leq q$$

$$\text{and } V(q, r) = V(q, r-1) + V(q, r-2) + \ldots\ldots + V(q, r-q), \text{ if } r > q\,.$$

The sequence has $(n + 1)$ terms. Here, we note that, if we take $q = 2$, we get the sequence 1, 1, 2, 3, 5, 8, 13, ..., which is the socalled Fibonacci numbers. Thus, Nārāyaṇa's *Sāmāsikā* sequence is a generalization of the socalled Fibonacci sequence.

Ex. 3 "O friend, tell quickly, how the *sāmāsikī* sequence looks, where the sum is 7 and the greatest digit is 3."

Obviously, here the *sāmāsikī* sequence will be 1, 1, 2, 4, 7, 13, 24, 44.

PĀTĀLA SEQUENCE

R. 15b–17a. "Write 0 and 1 below the *sāmāsikā* sequence. Add the sum of the numbers written below in places equal to 'the greatest digit' from the reverse order to the number

(of the *sāmāsikā* sequence)' just above the last number written below (and) write (the sum) ahead of that (i.e., ahead of the last number written below). (Repeat the process for) all terms (till the number of numbers, written below, becomes equal to that of the *sāmāsikā* sequence). If the numbers (written below) in places equal to the greatest digit be not available, take the sum of numbers in such places as are available. (Numbers written below form the *pātāla* sequence)."

As we will see later on, Nārāyaṇa has used his *pātāla* sequence also in partitions of a number n. So, here also q and n have the same meanings as attached to them in the previous rule (R. 13–15a). Let $V(q, r)$ and $w(q, r)$ denote the rth term of the *sāmāsikā* sequence and the *pātāla* sequence, respectively, where q is the greatest digit used in the partitions of a number n.

According to the rule,

$w(q, 1) = 0, w(q, 2) = 1$

$w(q, r) = V(q, r-1) + w(q, r-1) + w(q, r-2) + \ldots\ldots + w(q, 2) + w(q, 1)$, if $3 \leq r \leq q$,

and $w(q, r) = V(q, r-1) + w(q, r-1) + w(q, r-2) + \ldots\ldots + w(q, r-q)$, if $r > q$.

The sequence has $(n + 1)$ terms.

Ex. 4. "O friend, if you know, tell how the *pātāla* sequence looks where the sum is 7 and the measure of the greatest digit is 3."

Obviously, here the *pātāla* sequence will be,

0, 1, 2, 5, 12, 26, 56, 118.

GEOMETRIC PROGRESSION

R. 17b–18a. "Write 1 in the beginning and then that (i.e., 1) multiplied by the greatest digit, ahead of that. Again, in the same way, (the numbers multiplied by the greatest digit written ahead) until (the number of terms is) 1 more than the number of places is called the Geometric Progression."

If q be the greatest digit and p, the number of places, according to the rule, the geometric progression will be, $1, q, q^2, \ldots\ldots, q^p$.

Ex. 5 "Friend, tell quickly, how the geometric progression looks when the greatest digit is 3 and the number of places is (also) 3."

Clearly, according to the rule, the geometric sequence will be 1, 3, 9, 27.

Nārāyaṇa deals with several types of sequences which are meant for their applications to different types of permutations and partitions etc. The number of places (i.e., of digits) in a permutation or a partition etc. or the greatest digit used in such a permutation or a partition etc., as the case may be, play an important role in the treatment of these topics. Therefore, here these sequences have been defined with respect to either the number of places or the greatest digit or both according to their use. This seems to be the reason behind the fact that Nārāyaṇa has used the word *śreḍhī* for progression in general cases (Chapter III) while he has used the work *paṅkti* for a sequence for their use in permutations or partitions, etc.

ABHYĀSIKĪ SEQUENCE

R. 18b–21a. "Multiply the number of places by the greatest digit. Increase the product by 1 (and then) subtract 'the number of places' from the sum. The remainder is the measure of the number of terms of the *abhyāsikī* sequence. Leaving the last number of 'the arithmetic progression of measure equal to the greatest digit', as it is, multiply the other numbers by the greatest digit and put (the products) ahead of that (i.e., ahead of the arithmetic progression). In this way, again and again, multiply those by 'the greatest digit' (and put the products ahead of the arithmetic progression) until the measure of their number is equal to the number of places'."

Let q be the greatest digit and p, the number of places. According to the rule, $p(q-1)+1$ terms of the sequence, $1, 2, 3, \ldots\ldots, q, 2q, 3q, \ldots\ldots, q^2, 2q^2, 3q^2, \ldots\ldots, q^3, 2q^3, 3q^3, \ldots\ldots$, will form the *abhyāsikī* sequence.

Ex. 6 "O friend, if you know concatenation of numbers, tell the *abhyāsikī* sequence when the greatest digit is 3 and the number of places is (also) 3."

Clearly, here the *abhyāsikī* sequence will be, 1, 2, 3, 6, 9, 18, 27.

SŪCĪ SEQUENCE OR *NĀRĀCIKĀ* SEQUENCE

(SEQUENCE OF POLYNOMIAL COEFFICIENTS)

R. 21b–22a. "Put the sequence of separate units (*vaiśleṣiṇī paṅkti*) of measure equal to the greatest digit, separately, in places equal to the number of places. Their (final) product is the *sūcī-paṅkti* or the *nārācikā-paṅkti* (sequence of polynomial coefficients:"

Let p be the number of places and q, the greatest digit. According to the rule, the sequence of unities of measure q is kept at p places, one below the other. Then, those kept at first 2 places are multiplied. The product sequence, so obtained, is multiplied by the sequence kept at the 3rd place and so on. Numbers of the final

product form the *sūcī* or the *nārācikā* sequence (i.e., the sequence of polynomial coefficients). Let $u(p, q, r)$ denote the rth terms of the sequence. Then $u(p, q, r)$ are the coefficients of x^r, $r = 0, 1, 2, \ldots, p(q-1)$, in the polynomial $(1 + x + x^2 + \ldots + x^{q-1})^p$.

Rationale : Let $u(p, q, r)$ denote the rth term of the *sūcī or nārācikā* sequence. The sequence of unities of measure q, i.e., 1, 1, 1, ..., 1 (the number of unities being q) are the coefficients of x^r, $r = 0, 1, \ldots, q-1$

in the polynomial

$(1 + x + x^2 \ldots + x^{q-1})^1$.

When another sequence of unities is kept one below the other and then multiplied, obviously, the product of the sequence, so obtained, will be the coefficients of x^r, $r = 0, 1, 2, \ldots, 2(q-1)$, in the polynomial

$$(1 + x + x^2 + \ldots + x^{q-1})^2$$

Proceeding in this way, the final product of the sequences obtained after multiplication of sequences of unities kept at p places will be the coefficients of x^r, $r = 0, 1, 2, \ldots, p(q-1)$, in the polynomial $(1 + x + x^2 + \ldots + x^{q-1})^p$.

Thus, $u(p, q, r)$ = coefficient of x^r, $r = 0, 1, 2, \ldots, p(q-1)$,
in the expansion of

$$(1 + x + x^2 + \ldots + x^{q-1})^p.$$

Here, we observe that if $q = 2$,
$u(p, 2, r)$ = coefficient of x^r ($r = 0, 1, 2, \ldots, p$) in the expansion of $(1 + x)^p = C(p, r)$.

Thus, Nārāyaṇa's sequence of polynomial coefficients is his generalization of the sequence of binomial coefficients.

Ex. 7 "O learned, if you know, tell how the *nārācikā* sequence (i.e., the sequence of polynomial coefficients) looks when 'the number of places' is 3 and 'the greatest digit' is (also) 3."

Obviously, the sequence of polynomial coefficients in this case will be 1, 3, 6, 7, 6, 3, 1. For, keeping the sequence of separated units, when $q = 3$, i.e., 1, 1, 1 at 3 places and then multiplying them step by step, we have,

					1,	1,	1	Ist place
					1,	1,	1	2nd place
					1,	1,	1,	
				1,	1,	1,		
			1,	1,	1			
			1,	2,	3,	2,	1	
					1,	1,	1	3rd place
			1,	2,	3,	2,	1	
		1,	2,	3,	2,	1		
	1,	2,	3,	2,	1			
1,	3,	6,	7,	6,	3,	1		

YAUGIKA SEQUENCE

R. 22b–23a. "Multiply 'the greatest digit' by 'the number of places'. (The product) is the first (sum). Subtract 1 from that, again and again, until a number equal to 'the number of places' is obtained. This sequence is called the *Yaugika* sequence."

Let q be 'the greatest digit' and p, 'the number of places'. According to the rule, $pq, pq - 1, pq - 2, \ldots, p$ is called the *Yaugika* sequence. The sequence is utilized in finding the number of permutations of digits from 1 upto q taking p at a time in which any digit may be repeatedly used in a permutation. The permutations are so arranged that the first permutation has the greatest sum which is obviously equal to pq. Hence the word 'first' (sum) in the definition of the sequence.

Ex. 8. "O learned, if you know, tell quickly, how the *Yaugika* sequence looks when the greatest digit is 3 and the number of places is (also) 3."

Obviously, here the *yaugika* sequence will be, 9, 8, 7, 6, 5, 4, 3.

PARTIAL *SŪCĪ* SEQUENCE (i.e. PARTIAL SEQUENCE OF POLYNOMIAL COEFFICIENTS)

R. 23b–25a. "Write the sequence of polynomial coefficients obtained from (the number of places equal to) 'the number of places less than 1' below the *yaugika* (sequence). In the absence of numbers (in the sequence of polynomial coefficients) write ciphers. Such number of numbers in the sequence of polynomial coefficients from below the stated sum on the reverse side (i.e., towards the left, from left to right) as are equal to 'the greatest digit' is called the Partial Sequence of Polynomial Coefficients."

For further explanation of the rule see the solution of the example given below.

Ex. 9. "Tell quickly, how the figures (i.e., numbers) of the partial sequence of polynomial coefficients look, when the number of places is 3, the greatest digit is (also) 3 and the sum of the digits is 6."

Here, $p = 3 = q$, so the *yaugika* sequence is, 9, 8, 7, 6, 5, 4, 3, and the sequences of polynomial coefficients of (3 – 1=) 2 places is 1, 2, 3, 2, 1. Writing them as suggested, we have,

9, 8, 7, 6, 5, 4, 3 　　　　1, 2, 3, 2, 1, 0, 0

According to the rule, if the sum be 7, 5, or 4, the partial sequence of polynomial coefficients will be, 1, 2, 3; 3, 2, 1 or 2, 1, 0, respectively.

FIGURES OF NUMBERS

KHAṆḌA MERU

R. 25b–28. "Form cells, (their number being) equal to the number of places, horizontally. Form the cells in rows (one row below the other leaving the first cell) in succession till the end. Write 1 in the first cell and ciphers, in the other cells of the first row. Write those numbers (which are) the (continued) products of the numbers of the arithmetic progression in the cells below that (i.e., below the first row, in order). In this way, moving forward, multiply the remaining products by the number of the row (below the first row, and write the resulting products in the lower successive rows). (Thus) the *Khaṇḍa Meru* is formed. The (serial) number (of a permutation, in the serial arrangement of all permutations) is determined by the sum of numbers in diagonal (i.e., oblique) cells (of the corresponding) *Khaṇḍa Meru.*"

Let the number of places be p. According to the rule, $p, p-1, p-2, \ldots,$ 3, 2, 1, cells are formed in the 1st, 2nd, 3rd, ..., pth rows, one row continuously below the other, in order. Cells are so formed that there are 1, 2, 3, ..., p, cells in the 1st, 2nd, 3rd, ..., pth columns, in order. 1 is kept in the 1st cell and ciphers in all other cells of the 1st row. In all other cells of the other rows, $n\underline{|m}$ *is kept in a cell which lies in the* $(m+1)th$ column of the $(n+1)th$ row.

Ex. 10. " O friend, if you know the process of concatenation of numbers, show how the *khaṇḍa meru* with the numbers looks, if the number of places be 6."

Here, the number of places = 6. So, the *khaṇḍa meru* formed according to the above rule will be as shown below :

1	0	0	0	0	0
	1	2	6	24	120
		4	12	48	240
			18	72	360
				96	480
					600

Fig. 1 (R. 25b–28, Ex. 10)

PATĀKĀ

R. 29–36a. "Numbers of the sequence of polynomial coefficients are the measures of the numbers of cells in the (corresponding) columns and the rows are determined by that. Numbers in the initial cells form the *pallava*. After that keep the numbers in the cells according to a plan. Write the *abhyāsikī* sequence in (the cells of) the first row. There, the last number is called the *Kṣepa* and that in its front is called the *Sādhya*. Add the former numbers to the *Kṣepa*, separately (and) in order (and) place those (i.e., the sums) in the lower row. *Patākā* is formed by the *sādhya*.

Suppose the *kṣepa* in the *sādhya* (i.e., suppose the next *sādhya* as the *kṣepa*). Add the *kṣepa* to the former numbers kept in the 1st, 2nd etc. cells (and) put (the sums) in cells of the row without numbers. The *kṣepa*, when added to a number lying below, (the sum) becomes a *sādhya*. Leaving that number, add the *kṣepa* to the other numbers. A *sādhya* is always in the geometric progression. (In this way) fill the cells again (and again) with numbers. Thus the *patākā* is formed."

Explanation : Let p be the number of places and q, the greatest digit. Also, let the rth term of the sequence of polynomial coefficients be $u(p, q, r)$, so that the sequence is

$$u(p, q, r), r = 0, 1, 2, \ldots, p(q-1).$$

Also, here the *abhyāsikī* sequence will be,

$$1, 2, 3, \ldots, q, 2q, 3q, \ldots, q^2, 2q^2, 3q^2, \ldots, q^3, \ldots, q^p.$$

Cells are formed continuously such that their numbers are, $u(p, q, r), r = 0, 1, 2, \ldots, p(q-1)$, in the 1st, 2nd, 3rd, ..., $(pq - p + 1)$th columns, in order. Cells are filled with

numbers, in steps. In the 1st step, numbers of the *abhyāsikī* sequence are filled in cells of the first row, in order.

In the 2nd step, keeping $(2q - 1)$, in a cell left of and next to the column in a cell of which $2q$ has been filled, numbers $2q - 1, 2q - 2, \ldots, q + 1$ are filled such that each succeeding number is filled in a cell of the column which is towards the left and is next to the column in which the preceding number has been filled. Similarly, the numbers $3q - 1, 3q - 2, \ldots, 2q + 1$, are filled in the cells. In this way, numbers upto $(q^2 - 1)$ are filled in the cells.

After that, the numbers, $2q^2 - q, 2q^2 - 2q, \ldots, q^2 + q$ are filled in cells by the process indicated in the 2nd step. Again numbers in between any two of the above numbers are filled in cells by the same process. The process is repeated for numbers in between $3q^2$ and $2q^2$ and then for $4q^2$ and $3q^2$ and so on. In this way, numbers upto q^3 are filled. The process is repeated, again and again and in this way, numbers upto q^p are filled.

The author of *Ratna Mañjūṣā* has[8] named such figure of numbers as *Meru Prastāra* and has[9] given the rule for its formation in detail. The author (commentator?) has[10] illustrated his rule with 2 figures of numbers, one for a *varṇavṛtta* of 4 *mātrās*, and the other, for that of 5 *mātrās*.

Dhvajā meaning thereby *patākā* i.e., a flag, has[11] been mentioned by Brahmagupta. Rules related to it have not been deciphered so far.

Ex. 11. "O friend, if you know, tell me quickly, how the *patākā* looks, where the greatest digit is 3 and the number of places is (also) 3."

In this case $p = 3 = q$. Nārāyaṇa gives the following figure of numbers as the solution of the above example and example 33 (R. 70).

9	8	7	6	5	4	3
1	2	3	6	9	18	27
	4	5	8	15	24	
	10	7	12	17	26	
		11	14	21		
		13	16	23		
		19	20	25		
			22			

Fig. 2 (R. 29–36a, Ex. 11 and R. 70, Ex. 33)

SUMERU

The *sumeru* is an extension of the concept of *meru*. Whereas, the numbers in cells of a row of the *meru* consist of terms of a sequence of binomial coefficients, the numbers in cells of a row of a *sumeru* consist of the products of terms of a sequence of binomial coefficients multiplied by the powers of a constant.

R. 36b–39. "Starting from 1 (cell) in the first (row), form cells increasing by 1 in (successive) rows one below the other (continuously) until the number (of cells in a row) is 1 more than the number of places. Write 1 in the 1st cell (of the 1st row). Write '1 less than the greatest digit' called *śṛṅga* in the 1st cell of the 2nd row and 1 in the other cell of (the 2nd row).

In (the case of) any other cell, multiply the number lying just above that cell by the *śṛṅga*. Add the product to the number in the cell, left of the cell vertically above (that cell) and put (the result) in the (lower) cell. In this way, (following) this process, fill the cells in rows with numbers, in order.

This method for the formation of *sumeru* has been told by the learned mathematicians."

Explanation : 1, 2, 3, ..., cells are formed in the 1st, 2nd, 3rd, ..., rows, in order, such that 1st cell in all the rows are in the 1st column, the 2nd cells, in the 2nd column and so on. Let q be the greatest digit and s, the *śṛṅga,* so that $s = (q - 1)$. 1 is written in the only cell of the 1st row. For filling any other cell with a number, the number, in the cell just above the cell to be filled, is multiplied by s and the number in the cell lying towards the left of the cell (whose number has been multiplied) is added to the product. The sum is filled in the desired cell. If the cell above a cell or left of a cell is absent, it is supposed that the cell contains a cypher.

Writing s for the *śṛṅga*, we observe that the following figure of numbers is formed for $p = 6$.

1						
s	1					
s^2	$2s$	1				
s^3	$3s^2$	$3s$	1			
s^4	$4s^3$	$6s^2$	$4s$	1		
s^5	$5s^4$	$10s^3$	$10s^2$	$5s$	1	
s^6	$6s^5$	$15s^4$	$20s^3$	$15s^2$	$6s$	1

Fig. 3, (R. 36b–39)

Thus, according to the rule, $(r + 1)$th row of the *sumeru* contains the expressions,

$$C(r, 0)s^r, C(r, 1)s^{r-1}, C(r, 2)s^{r-2}, \ldots, C(r, r-1)s, C(r, r) \text{ where } s = q - 1.$$

Here, we observe that *sumeru* is a generalization of the *meru prastāra* as given by earlier authorities on Sanskrit prosody. For, if we take $q = 2$, so that $s = 1$, the *sumeru* becomes the *meru prastāra*. The origin and studies of the properties of *meru prastāra* has been dealt with later on in introduction to R. 65–66.

Ex. 12. "O learned, if you know, (tell) here, how the *sumeru* looks, if the greatest digit is 3 and the number of places is (also) 3."

Here, $q = 3$, so $s(= q - 1) = 2$. Also, $p = 3$. Therefore, the *sumeru* will be as given below :

1			
2	1		
4	4	1	
8	12	6	1

Fig. 4. (R. 36b–39. Ex. 12)

MATSYA-MERU

The *matsya-meru* is an extension of the concept of *meru* in another way. Whereas the numbers in the cells of a row in a *meru prastāra* consist of terms of the sequence of binomial coefficients, the numbers in the cell of a row in a *matsya-meru* consist of terms of the sequence of polynomial coefficients.

R. 40–43. "(Form) one (cell) initially (and then), row of cells, (such that the number of cells in a row) increases by '1 less that the greatest digit' until the measure of that is '1 more (than the sum of digits'). Leaving the 1st cell and starting below the 2nd, form (continuous) rows (of cells) equal to the sum of digits.

After that put 1 in (each cell) of the 1st 2 rows. (Take cells) above one's own cell and left of it, in the reverse order and equal to the greatest digit. Write the sum of numbers (in them) in the lower cell. If there be an absence (of cells) equal to the greatest digit, (take) the sum of such numbers (in cells) which are possible. Work, in order. "

Explanation : Let the number of places be p, the greatest digit, q and the sum of digits, n. 1 cell is formed in the 1st row. Leaving a cell below the above cell, q cells are formed in the 2nd row. After that, starting below the 2nd cell of the row lying above, $(n - 1)$ continuous rows of cells are formed such that their

number is $r(q-1)+1$ in $(r+1)$th row. Thus, the total number of rows is $(n+1)$.

1 is written in each cell of the 1st two rows. For filling any other cell with a number, q cells from right to left and lying towards the left of that cell in the row lying above are chosen. The numbers in these cells are added and written in the cell lying below. Where the numbers of available cells in the row lying above and towards the left of a cell is less than q, the number (or the sum of numbers) in a cell (or cells) as are available is written in the cell lying below.

Ex. 13. "O learned, tell the *matsya-meru* where the greatest digit is 3, the number of places is (also) 3 and the sum of digits is 7."

Here, $p = 3$, $q = 3$ and $n = 7$.

The figure of numbers, formed by the above rule, will be as shown below.

1									
	1	1	1						
		1	2	3	2	1			
			1	3	6	7	6	3	1
				1	4	10	16		
					1	5	15		
						1	6		
							1		

Fig.5, (R. 40–43, Ex. 13)

The figure of *matsya-meru* given in the printed text is vitiated, for it does not satisfy the rules. The figure shown here is based upon a manuscript of the *Gaṇita Kaumudī* available at Sampurnanand Sanskrit University, Varanasi (Manuscript No. 35668/p–7 recto)

ANOTHER METHOD FOR THE FORMATION OF POLYNOMIAL COEFFICIENTS

R. 44a–44b(i). "Numbers in cells of a row (of the *matsya-meru*) form the sequence of polynomial coefficients consistent with the number of places."

Explanation : According to the rule, numbers in any row of the *matsya-meru* form the sequence of polynomial coefficients consistent with the number of places. Thus, if q be the greatest digit, numbers in $(p+1)$th row of the *matsya-meru*

are the coefficients of x^r for $r = 0, 1, 2, \ldots, p(q-1)$, in order, in the expansion of $(1 + x + x^2 + \ldots + x^{q-1})^p$.

Initially, it seems that this process of formation of the sequence of polynomial coefficients is different from that given in R. 21b–22a, since the multiplication part of the process in that rule is absent in this rule. However, basically both the processes are the same. This is so, because initially in both the cases we start with the sequence of separated units. Moreover, in this case, the number of cells in each subsequent lower row, increases by $(q-1)$ which gives $(q-1)$ more numbers in each subsequent lower row. This will be amply clear by comparing the formation of the sequence by R. 21b–22a with that given in this rule, specially by comparing the solutions of Ex. 7 with that of Ex. 13.

R. 44b(ii)–c(i). "Separately, their sums (i.e., sums of numbers in different rows) form a geometric progression."

According to the rule, if numbers in cells of different rows are added separately, they form a geometric progression.

Rationale : This will be so because 1 is in the cell of the 1st row. The sequence of separated units of measure q is in cells of the 2nd row, so their sum is q. Each subsequent sequence of polynomial coefficients is obtained by multiplication from the previous one by the sequence of separated units whose sum is q. So, obviously, sum of numbers in cells of different rows will form the geometric progression $1, q, q^2, \ldots, q^p$.

Just as binomial coefficients were fore-runner of the binomial theorem so, like that polynomial coefficients were fore-runner of the polynomial theorem. Now, if we compare this rule in which Nārāyaṇa has given a property of the polynomial coefficients, with R. 67b given later on in which he states the binomial theorem, inversely, we observe that Nārāyaṇa while givig this rule, remarkably anticipates the polynomial theorem in the form,

$$(1 + x + x^2 + \ldots + x^{q-1})^p = \sum_{r=0}^{(q-1)p} u(p, q, r) \cdot x^r$$

Obviously, if $x = 1$, the above rule is obtained.

RELATION BETWEEN *SĀMĀSIKĀ* SEQUENCE AND POLYNOMIAL COEFFICIENTS

We have seen earlier that *sāmāsikā* sequence is a generalization of the socalled Fibonacci sequence (note after R. 13–15a. We have also seen that the sequence of polynomial coefficients is a generalization of the sequence of binomial coefficients (note after R. 21b–22a). The author of *Prākṛta Paiṅgala*[12] established a relation

between the socalled Fibonacci numbers and the binomial coefficients so that the former can be expressed in terms of the latter.

Just like the *meru* (or *prāsāda*) *prastāra* (i.e., the socalled Pascal's triangle) the author of *Prākṛta Paiṅgala* gives a rule for the formation of a figure of numbers called[13] *Mattā-Meru* (i.e., *Mātrā Meru*) by him. With the help of *Mattā Meru* he establishes a relation between the binomial coefficients and the socalled Fibonacci numbers. If fact, according to it, a socalled Fibonacci number is the[14] sum of certain binomial coefficients and thus the author gives another method for the formation of the former. Starting from 1, let t_n denote the nth socalled Fibonacci number. The book contains[15] the rule, which can be expressed as

$$t_n = \sum_{r=0}^{m} C(n-r, r) \quad \text{where } m = n/2, \text{ or } (n-1)/2$$

according as n is odd or even.

Nārāyaṇa extends this rule to his above two generalized sequences. He establishes a relation between the *sāmāsikā* sequence and the sequence of polynomial coefficients so that the former can be expressed in terms of the latter.

R. 44c(ii)–45a "Sums of numbers in different columns of that (i.e., of the *matsya-meru*) form a *sāmāsikā* sequence."

Explanation : We have seen earlier that numbers in cells of the *matsya-meru* are polynomial coefficients (R. 21b–22a). According to this rule, the sums of polynomial coefficients in cells of columns of the *matsya-meru* form a *sāmāsikā* sequence.

Let $V(q, r)$ and $u(p, q, r)$ denote the rth term of the *sāmāsikā* sequence and the sequence of polynomial coefficients, in order, where p and q have their usual meanings. By R. 13–15a and R. 21b–22a, we have,

$$V(q, 1) = 1 = u(0, q, 0)$$

$$V(q, 2) = 1 = u(1, q, 0)$$

For other values of p and q, we observe that

$$V(q, 3) = u(2, q, 0) + u(1, q, 1)$$

$$V(q, 4) = u(3, q, 0) + u(2, q, 1) + u(1, q, 2)$$

and so on.

In general, $V(q, t+1) = u(t, q, 0) + u(t-1, q, 1) + \ldots\ldots\ldots\ldots + u(t-k, q, k)$ where, $k = p + r - t$ and $t \le (p+r)q < t+1$.

Rationale : A comparison of these relations with the *matsya-meru* (R. 40–43, when the greatest digit is q) shows that the right side corresponding to the tth relation are the numbers in cells of the tth column of the *matsya-meru* whose sum gives the tth term of the *sāmāsikā* sequence and hence the rule.

PRATYAYA

Pratyaya in Sanskrit means analysis, basis, definition, proof, notion and conception etc. According to Mahāvīra, these in relation to Sanskrit metres are (1) *Prastāra*, (2) *Saṅkhyā*, (3) *Naṣṭa*, (4) *Uddiṣṭa*, (5) *Lagakriyā*, and (6) *Adhvāna*. A knowledge of these were necessary not only for a good understanding of Sanskrit prosody but also for its development. *Prastāra, saṅkhyā, naṣṭa and uddiṣṭa* etc. with regard to a metre has been described[16] by Brahmagupta as well, though the chapter on prosody in his *Brāhmasphuṭa Siddhānta* has not been translated, so far.

Nārāyaṇa has extended and generalized these *pratyayas* of Sanskrit prosody to Combinatorics. Besides dealing with the 1st 5 of these, he has also described some other *pratyayas* such as *ūrdhva-paṅkti-yoga* (i.e., the sum of digits at a particular place, in all permutations), *āvṛtti* (i.e., the number of permutations having a particular digit at a particular place, specially the number of those ending in a particular digit), *sarva-yoga* (i.e., the sum of all permutations) and *aṅka-pāta* (i.e., total number of digits in all permutations). Most of these *pratyayas* have been described for the partitions of a number as well.

PERMUTATIONS OF n DIFFERENT DIGITS (OR THINGS) TAKING ALL AT A TIME

(a) Total number of permutations;

(b) Number of repetitions of a digit at a particular place;

(c) Sum of digits added vertically at a particular place;

(d) Sum of all numbers (so) formed by permutations; and

(e) Total number of digits in all permutations.

R. 45b–48 "In dissimilar digits, the product of (numbers in) the arithmetic progression happens to be the total number of permutations (*saṅkhyā-bheda*[17]). In case of idols with dissimilar weapons (in hands) also, one should know the same. Multiply the number of permutations by the sequence of augmented units (and) leave the last product. The others, divided by the number of places, happen to be the number of permutations ending in a particular digit (*antya-bheda*), starting from the lowest etc.

(onwards) in the reverse order. Multiply the number of permutations ending in a particular digit by that digit. All such products when added vertically, become the sum of digits in a particular place in all permutations (*ūrdhva-yoga*[18]). This multiplied by the line of unseparated units happens to be the sum of all permutations (*sarva-yoga*[19]).

The number of permutations ending in a particular digit multiplied by the number of places give the total number of that digit (in all permutations). Their sum is the total number of digits in all permutations. In the case of *jalaukā* sequence also, this one is the process."

Let there be n different digits $d_1, d_2, d_3, \ldots\ldots, d_n$. According to the rule :

(a) Total number of such permutations = $\lfloor\underline{n}$;

(b) Number of repetitions of a digit at a particular place = $\lfloor\underline{n-1}$;

(c) Sum of digits added vertically at a particular place
$= (d_1 + d_2 + \ldots\ldots + d_n).\lfloor\underline{n-1}$.

(d) The sum of all the numbers formed by the permutations
= (sum of digits). (line of unseparated units of n places). $\lfloor\underline{n-1}$.

(e) Total number of digits in all permutations = $n\lfloor\underline{n}$;

Rationale : (a) By the process of arrangement of n digits, $d_1, d_2, \ldots\ldots, d_n$, as given by Nārāyaṇa, which has been given later on (R. 49–51a), we observe that the 1st place can be filled in n ways as any one of the n digits can be kept in this place. Now in each case of filling the 1st place, the 2nd place can be filled in $(n-1)$ ways since after filling the 1st place only $(n-1)$ digits are left and any one of these can be kept in the 2nd place. So, the 1st two places can be filled in $n(n-1)$ ways and so on. Now, since there are n places and n digits, total number of permutations $n(n-1)(n-2)\ldots\ldots 2.1 = \lfloor\underline{n}$.

(b) There are n digits and the number of permutations ending in each digit will be the same, due to symmetry and so the number of permutations ending in any particular digit $= \dfrac{\lfloor\underline{n}}{n} = \lfloor\underline{n-1}$.

(c) As there are n digits, $d_1, d_2, \ldots\ldots d_n$, and the number of permutations ending in any one of these is $\lfloor\underline{n-1}$, so due to symmetry, the number of permutations such that any particular digit occupy any particular place will also be the same. Therefore, the sum of digits at any particular place will, obviously, be equal to

$(d_1 + d_2 + \ldots\ldots + d_n)\lfloor\underline{n-1}$ = (sum of digits). $\lfloor\underline{n-1}$.

(d) Digits are on unit's place or on ten's place or on hundred's place and so on upto the digits on 10^{n-1} place. So, the sum of all permutations

= (the sum of digits at a particular place in all permutations) ×

$$(10^{n-1} + 10^{n-2} +, \ldots\ldots + 10^2 + 10 + 1)$$

= (sum of digits). $\lfloor\underline{n-1}$. (line of unseparate units of n places),

from above and R. 10.

(e) Since the number of places is n, each permutation has n digits. Also, the number of permutations is $\lfloor\underline{n}$. So, the number of digits in all permutations $= n.\lfloor\underline{n}$

The earliest Chinese example on combinatorics occurs[20] in the famous Chinese book of changes or *I Ching*. This seems[21] to be a compilation of material which dates from the 7th century B.C. The system of *I Ching* was based upon two symbols, the Yang (—) and Yin (– –). These were combined into trigrams and hexagrams. There are $2^3 = 8$ trigrams and $2^6 = 8$ hexagrams and in describing them the ancient Chinese verified the n^r rule for combinations with repetitions. Rabbi ben Ezra (1140 A.D.) though states[22] no general law, he seems to be aware of the rule for finding the combinations of n things taking r at a time. Boethius (510 A.D.) gives[23] a rule for finding the combinations of n things taking 2 at a time. Levi ben Gerson (1321 A.D.) gave rule[24] for the combination of n things taking r at a time and rules for the permutations of n things taking all together and also for those taking r at a time.

Ex. 14 "O friend, if you have laboured in 'concatenation of numbers', tell separately, the number of permutations, the number of permutations ending in (or) starting with a particular digit, the sum of digits in a particular place, the sum, and total number of digits, in all permutations when the digits are (either) 7, 3, 9 (or) 1, 6, 3, 2 (or) 1 upto 9."

Ex. 15 "An idol of Śiva has a bow, an arrow, a *khaḍga* (a sharp iron instrument), a *damarū* (a kind of musical instrument), a *kakapāla*, a noose, a *khaṭvāṅga* (a club with a skull at the top), a *śūla* (a sharp iron stake), a snake, and a *śakti* (a spear or a lance) in his different hands and (an idol of) Hari has a lotus, a club, a *cakra* (a sharp circular missile weapon) and a conch shell in his hands. By changing instruments in their hands how many idols can be formed."

EXPANSION (*PRASTĀRA PRATYAYA*) : SERIAL ARRANGEMENT OF PERMUTATIONS *PRASTĀRA PRATYAYA*

In Sanskrit prosody *prastāra* means 'spreading out' so as the exhibit and usually the term is understood as referring to 'the spreading out of various forms which a metre may assume'. From combinatorics point of view the idea is to determine the configurations under given constraints.

R. 49–51(a) "(The order) in which the smaller amongst the digits is kept earlier should be known to be 'in order'. Starting from the beginning (each time), keep (the next) lesser (digit) below the greater (and) the rest (on the right) as above (and the others on the left), in order, until (the digits), in order, become in the reverse order. This is the method of expansion of digits, idols, sounds of music and musical instruments."

Explanation : Starting from the beginning each time, the 1st two digits which are in ascending order, are found out. The digits on the right of the greater digit are written as they are, the greater digit is replaced by the smaller digit and the other digits are written, on the left of the digit which has replaced the greater digit, in ascending order. The process is continued until all the digits become in descending order.

Ex. 16 "O friend, if you have acquired proficiency in 'the concatenation of numbers' tell the expansion of 1st and 2nd cases in the example (stated earlier)."

Nārāyaṇa gives the expansions of Exs. 14 (1st 2 cases only) and 15. Consider the expansion of the 2nd case of Ex. 14.

The expansion is —

(1) 1236	(7) 1263	(13) 1362	(19) 2361
(2) 2136	(8) 2163	(14) 3162	(20) 3261
(3) 1326	(9) 1623	(15) 1632	(21) 2631
(4) 3126	(10) 6123	(16) 6132	(22) 6231
(5) 2316	(11) 2613	(17) 3612	(23) 3621
(6) 3216	(12) 6213	(18) 6312	(24) 6321.

By R. 45–48,

(a) Total number of permutation (*saṅkhyā-bheda*) (= $\lfloor\underline{4}$) = 24

(b) Number of repetitions of a digit at a particular place (*āvṛtti*) (= $\lfloor\underline{4-1}$) = 6.

(c) Sum of digits, added vertically, at a particular place (*urdhva–yoga*)
[= (1 + 2 + 3 + 6) . $\lfloor\underline{4-1}$] = 72.

(d) Sum of all the numbers so formed (*sarva-yoga*)
[= (1 + 2 + 3 + 6). 1111. $\lfloor\underline{4-1}$] = 79992.

(e) Total number of digits in all the permutations (*aṅka pāta*) = 4 $\lfloor\underline{4}$ = 96.

Ex. 17 "O friend, if you have the ability to cross the ocean of 'concatenation of numbers' tell how the expansion of variations (i.e. permutations of *gadā* etc. in hands) of idols of Murāri looks."

NAṢṬA (LOST) AND *UDDIṢṬA* (DESIRED) *PRATYAYAS*

Naṣṭa bheda (or *uddiṣṭa bheda*) is the unknown (or desired) particular variation (or permutation etc.) whose *naṣṭāṅka* (or *uddiṣṭaṅka*), i.e., whose serial number in the serial arrangement of all variations (or permutations) under given restraints is known and vice-versa.

SERIAL NUMBER OF A GIVEN PERMUTATION

R. 51b–53 "Form *Khaṇḍa-meru* whose measure is the given number of places. Count the last digit in the given arrangement in the arrangement of the digits in ascending order, from the end (leftwards and) encircle the number in the cell which has the same number of placing (when counted vertically downwards) in the last vertical column of the *khaṇḍa-meru*. Remove that digit from the given arrangement (and) the digits in ascending order, both. Repeat the process, again and again, until the remaining digits (in the given arrangement) and those in (the ascending) order become the same. Add the sum of numbers in the encircled cells (of the *khaṇḍa-meru*) to that in its initial cell (i.e., 1). The sum is the measure of the serial number of the given arrangement (in the serial arrangement of all permutations with the given digits)."

Explanation : Let the number of places be r and let a *khaṇḍa-meru* corresponding to r places be formed by R. 25b–28. The digit occupying the last place in the given arrangement is counted from the end, leftwards, in the arrangement of the digits in ascending order. Suppose, we get, m. Then counting vertically downwards, the mth cell in the last vertical column of the *khaṇḍa-meru* is marked and the last digit is removed from the digits in ascending order. The process is repeated with the digit which now occupies the last but 1 place in the given arrangement.

If it is the kth digit amongst the digits in ascending order when counted (after removal of a digit/digits) from right to left starting from the end, the number in the kth (vertically downwards) cell in the last but 1 column of the *khaṇḍa-meru* is marked and the digit is removed from the digits in ascending order.

The process is repeated, again and again, until digits in the given arrangement after successive removals of the digits become in ascending order. The sum of numbers in the marked cells is added to 1. The result gives the desired serial number.

Rationale : Let there be n different digits $d_1 < d_2 < \ldots\ldots < d_n$ so that the digits in ascending order are $d_1 d_2 \ldots\ldots d_n$. Consider the last digit, say d_r, of the given permutation whose serial number is to be found out. Let us count d_r in $d_1\, d_2 \ldots\ldots d_n$ from right to left. We will get $n, n-1, \ldots\ldots,$ according as d_r is d_1, $d_2, \ldots\ldots,$ in order. Now consider the last column of the *khaṇḍa-meru*, nth,

$(n-1)$th, ... cells of this column contain the numbers, $(n-1)\,.\,\underline{\lfloor n-1}$, $(n-1).\underline{\lfloor n-2}$ . . . . . , in order. Also, d_1 ,d_2, ...will occupy the last position after $(n-1)$. $\underline{\lfloor n-1}$, $(n-1).\underline{\lfloor n-2}$, ..., permutations, in order. The rule is based on these 2 identical relations. Again, d_r is removed from the digits in ascending order and the process is continued with the digit which now occupies the last but 1 place, in the given permutation. Again, we get a similar result ue to the same reason. The process is continued till the end and hence the rule.

Ex. 18 "O mathematician, tell me here quickly the serial number of the arrangement 6231 (and that of the idol Viṣṇu) with a conch shell, a club, a *cakra* (a sharp circular missible weapon) and a lotus, if the original order for (the idol of) Viṣṇu be with a lotus, a club, a *cakra* and a conch shell."

Consider the 1st part of this example. There are 4 digits (i.e., the number of places) and so the *khaṇḍa-meru*, when the number of places is 4 will be as given below.

1	0	0	0
	1	2	6
		4	12
			18

Fig. 6, R. 51b–53, Ex. 18

The given permutation is 6231 and the digits in ascending order are 1236. Counting the last digit 1 of the given permutation amongst the digits in ascending order, we get 4. Fourth cell of the last column of the *khaṇḍa-meru* contains 18 and so we get 18.

Now 1 is removed from the ascending order. Last but 1 digit in the given permutation is 3 which when counted amongst the digits in ascending order, we get 2. Last but 1 column of the *khaṇḍa-meru* in its 2nd cell contains 2 and so, we get 2.

Similarly, from 2 of the given permutation, we get 1. Finally, the digit left is in ascending order and so, we get 1. The sum of the numbers, above obtained, is (18 + 2 + 1 + 1 =) 22 which is the desired serial number of the given permutation in the serial arrangement of all permutations with the given digits.

Permutation Corresponding To A Given Serial Number

R. 54–55a "Keep clods in cells of the (corresponding) *khaṇḍa-meru* such that the sum of numbers in them added to the initial number (in the *khaṇḍa-meru*, i.e., 1) becomes equal to the given serial number. The measure of the place (i.e., the cell) containing the clod happens to be (the measure of the digit for) the (desired permutation amongst digits) in ascending order. In this way the desired permutation is obtained."

Let there be n digits. *Khaṇḍa-meru* for the number of places, n, is formed. Taking 1 cell in each column of the *khaṇḍa-meru*, cells in the *khaṇḍa-meru* are marked such that the sum of numbers in them is equal to the given serial number. Now, if the number is in the mth cell of the nth column of the *khaṇḍa-meru*, the nth digit in the desired permutation is the mth digit, from right to left, in the digits kept in ascending order. The digit, so obtained, is kept separately and it is removed from the digits kept in ascending order. Next, the number in the $(n-1)$th column marked earlier is considered. The above process is followed with the remaining digits kept in ascending order and $(n-1)$th digit of the desired permutation is obtained. The process is repeated, again and again, until the desired permutation is obtained. The rule is converse of the rule given in R. 51b–53.

Ex. 19 "If you know, tell how the 15th permutation and the 11th idol of Viṣṇu of the previous example look."

Here, there are 4 digits, so, the number of places = 4. Therefore, the *khaṇḍa-meru* in this case will be as given earlier (Ex. 18). Considering numbers of the *khaṇḍa-meru* only, we have, 15 = 12 + 2 + 0 + 1 and the digits in ascending order are 1236.

12 is in the 3rd vertically downwards cell of the last column of the *khaṇḍa-meru*. So, last digit of the desired permutation will be the 3rd digit, amongst digits in ascending order, when counted from right to left from the end, i.e., 2. Now, 2 is removed from the digits in ascending order. Again, 2 is in the 2nd vertically downwards cell of the last but 1 column of the *khanda-meru*. So, the last but 1 digit of the desired permutation will be the 2nd digit amongst the remaining digits in ascending order, when counted from right to left from the end., i.e., 3. Proceeding further, in the same manner, we get 6 and then 1. Therefore, the desired permutation is 1632.

Permutations Of Digits (Or Things) Taking All At A Time When The Digits (Or Things) Are Not Different

(a) Number of permutations;

(b) Sum of digits in a particular place in all permutations;

(c) Number of permutations ending in a particular digit;

(d) Number of digits in all permutations; and

(e) Sum of all permutations.

R. 55b–56 "Divide total number of permutations (considering all the digits as dissimilar) by the number of permutations of (successive) equal digits (considering them as dissimilar). (The quotient) is the number of permutations[25]. Total number of digits is obtained by the method stated earlier. The rest (such as) the sum of digits in a particular place in all permutations etc. should be obtained with the help of *Jalauka* sequence[26]."

Explanation : Let there be n digits (so that the number of places $= n$) such that $d_1 < d_2 < d_3 < \ldots\ldots < d_t$. Let d_1 occur p times, d_2, q times, d_3, r times and all the rest be different.

According to the rule,

(i) The number of permutations $= \dfrac{\lfloor n}{\lfloor p \cdot \lfloor q \cdot \lfloor r} = N$ (say).

(ii) The number of digits in all permutations $= N.n$.

(iii) Rest of the results should be obtained with the help of *Jalauka* sequence. Thus, in this case, the sequence of augmented units will be, 1, 1, ..., 1 [the number of unities being $(n + 1)$], by R. 10b-11, and so the *Jalauka* sequence will be, 1, 1, ..., 1, r, q, p, 1 by R. 12. Now, according to this rule combined with R. 46-48.

(iv) The number of permutations ending in $d_1, d_2, d_3, d_4, \ldots\ldots, d_t$ will be

$$\frac{N}{n}p,\ \frac{N}{n}q,\ \frac{N}{n}r,\ \frac{N}{n},\ \ldots\ldots\ \frac{N}{n}, \text{ respectively.}$$

(v) The sum of digits in a particular place will be

$$\frac{Np}{n}\cdot d_1 + \frac{Nq}{n}\cdot d_2 + \frac{Nr}{n}\cdot d_3 + \frac{N}{n}\cdot d_4 + \ldots\ldots + \frac{N}{n}\cdot d_t\,.$$

(vi) The sum of all permutations will be

$$\left[\frac{Np}{n}\cdot d_1 + \frac{Nq}{n}\cdot d_2 + \frac{Nr}{n}\cdot d_3 + \frac{N}{n}\cdot d_4 + \ldots\ldots + \frac{N}{n}\cdot d_t\right]\cdot(11\ \ldots\ldots\ 1),$$

the number of unities in the latter factor being n.

Rationale : (i) Since d_1 occurs p times, d_2, q times, d_3, r times and all the rest digits are different and the number of digits are n, obviously,

the number of permutations $= \dfrac{\lfloor n}{\lfloor p \cdot \lfloor q \cdot \lfloor r} = N$.

(ii) Also, since there are n digits and the number of permutations is N and also all the digits are taken in each permutation, clearly,

total number of digits in all permutations = $N.n$.

(iii) Since d_1 occurs p times, the number of permutations ending in d_1

$$= \frac{\lfloor n-1}{\lfloor p-1 \cdot \lfloor q \cdot \lfloor r} = \frac{\lfloor n}{\lfloor p \cdot \lfloor q \cdot \lfloor r} \cdot \frac{p}{n} = \frac{N \cdot p}{n}.$$

Similarly, the permutations ending in $d_2, d_3, d_4, \ldots\ldots, d_t$ will be

$$\frac{Nq}{n}, \frac{Nr}{n}, \frac{N}{n}, \ldots\ldots \frac{N}{n}.$$

(iv) From (iii) it is clear that the sum of digits at the nth (i.e. the last) place will be,

$$(Np/n)d_1 + (Nq/n)d_2 + (Nr/n)d_3 + (N/n)d_4 + \ldots\ldots + (N/n)d_t$$

and so due to symmetry, the same will be the sum of digits at any particular place in all permutations.

(v) Also due to symmetry, the sum of digits in a particular place in all permutations

= the sum of digits occuring in units place

$$= \frac{N \cdot p \cdot d_1}{n} + \frac{N \cdot q \cdot d_2}{n} + \frac{N \cdot r \cdot d_3}{n} + \frac{N \cdot d_4}{n} + \ldots\ldots + \frac{N \cdot d_t}{n}$$

= the sum of digits occuring in tens' place

= the sum of digits occuring in hundreds' place

= . . . . = ... = ...

Therefore, the sum of all permutations

$$= \left(\frac{Npd_1}{n} + \frac{Nqd_2}{n} + \frac{Nrd_3}{n} + \frac{Nd_4}{n} + \ldots\ldots + \frac{Nd_t}{n}\right) . \left(10^{n-1} + 10^{n-2} + \ldots\ldots + 10^2 + 10 + 1\right)$$

$$= \left(\frac{Npd_1}{n} + \frac{Nqd_2}{n} + \frac{Nrd_3}{n} + \frac{Nd_4}{n} + \ldots\ldots + \frac{Nd_t}{n}\right) . (11 \ldots\ldots 1).$$

the number of unities in the last factor being n.

Ex. 20 "O learned mathematician, if you know how to be free from the concatenation of numbers, tell here, the number of permutations, their sum (etc.) from 2, 4, 1, 1 ; 5, 4 , 5, 4, 5 (and) 3, 3, 3, 9, 3."

EXPANSION (i.e., SERIAL ARRANGEMENT OF PERMUTATIONS)

R. 57 "The process for serial arrangement of permutations when the digits are not all different, is like that when the digits are all different. (Starting from the beginning each time), a greater digit is replaced by the lesser digit, (digits on the right are written as

above) and the remaining digits are written (on the left) in ascending order (until the digits become in descending order)."

Ex. 21 "O friend, if you have pride for the knowledge to ride the mountain of mathematics, tell how the serial arrangement of expansion of 1, 1, 2, 4 looks."

By the above rule, the author gives the expansions of 1, 1, 2, 4; 4, 4, 5, 5, 5 and 3, 3, 3, 3, 9.

The expansion of 1, 1, 2, 4 is as follows :

(1) 1124	(4) 1142	(7) 1241	(10) 4121
(2) 1214	(5) 1412	(8) 2141	(11) 2411
(3) 2114	(6) 4112	(9) 1421	(12) 4211.

COMBINATION OF n DISSIMILAR DIGITS (OR THINGS) TAKING r AT A TIME

Introduction : Studies of combination of dissimilar digits (or things) and their properties began very early in India. Solution of Ex. 22 given below (i.e., to find the value of $C(n, r)$ for $r = 1, 2, \ldots, 6$) and of $\sum_{r=1}^{6} C(n, r)$ both for $n = 6$ is found in *Caraka Saṁhitā*[27] and *Suśruta Saṁhitā*[28]. Discussions of some other examples such as those involving combinations of philosophical categories of senses and of eunuchs occur[29] in the work of the Jainas (2nd century B.C.). Systematic studies of rules to find the value of $C(n, r)$ and its properties occur in the works on Sanskrit and Prakrit prosody which will be described later on, in brief.

VALUE OF $C(n,r)$

R. 58 "(Numbers of) the arithmetic progression having the same number of places as are the digits (or things) are the divisors (and the same), in the reverse order, are the dividends. Multiply successively, the succeeding one by the preceding (quotient). (The products) determine the number of combinations taking 1 etc. at a time."

Let the number of digits or things be n, then the arithmetic progression will be $1, 2, \ldots, n$.

$n, n-1, \ldots, 2, 1$ are divided by $1, 2, \ldots, (n-1), n$, in order. Thus, we get, $\frac{n}{1}, \frac{n-1}{2}, \ldots, \frac{2}{n-1}, \frac{1}{n}$. Multiplying successively the succeeding quotient by the preceding ones, we get,

$$\frac{n}{1}, \frac{n(n-1)}{1.2}, \frac{n(n-1)(n-2)}{1.2.3}, \ldots, \frac{n(n-1)\ldots 2.1}{1.2\ldots(n-1)n}.$$

According to the rule, these are the combinations of n different digits (or things) taking 1, 2, ..., $(n-1)$, n of them at a time, in order.

The rule has been given[30] by Mahāvīra, Śrīdhara, Āryabhaṭa II and Bhāskara II. The rule, as given by some earlier authority has[31] been quoted by Bhaṭṭotpala (966 A.D.).

ALTERNATIVE RULE

R. 59a "Alternatively, leaving the 1st number, numbers of the *nārācikā* sequence (i.e., the sequence of polynomial coefficients) with 2 as the greatest digit (and with the same number of places determine the numbers of combinations taking 1 etc. at a time)."

According to the rule,
$u(n, 2, r) = C(n, r), r = 1, 2, \ldots\ldots, n.$

Rationale : From the law of formation of the sequence of polynomial coefficients, as given by Nārāyaṇa, we have seen earlier that,
$u(n, q, r) =$ coefficient of x^r, $r = 0, 1, 2, \ldots\ldots, n(q-1)$,
in the expansion of
$(1 + x + x^2 + \ldots\ldots + x^{q-1})^n$.
So, $u(n, 2, r) =$ coefficient of x^r, $r = 0, 1, 2, \ldots\ldots, n$
in the expansion of $(1 + x)^n$.
Nārāyaṇa has omitted to take $x = 0$, for obvious reasons. Therefore,
$u(n, 2, r) = C(n, r), r = 1, 2, \ldots\ldots, n.$

Ex. 22 "O friend, if you have good intelligence, tell, how many varieties of sauce with 6 flavours, basic, acidic, bitter, pungent, astringent and sweet, take 1 etc. (of the flavours at a time), can be prepared. (Also, tell them separately)."

We have seen earlier that the problem is a very old one. The problem and its solutions occur even in *Caraka Saṁhitā and Suśruta Saṁhitā*. Later on the problem has also been described[32] by Mahāvīra and Śrīdhara.

Ex. 23 "O eminent mathematician, if you are an expert in 'the concatenation of numbers' tell how many varieties of garlands can be prepared for the worship of Śiva with the flowers of green, red, white, black and blue colours, taking flowers of 1 colour etc. at a time. (Also, tell them) separately."

Ex. 24 "With the juice of orange, banana, mango, rose-apple, grape, sugarcane and *kharjura*, tell how many varieties of beverage can be prepared for the taste of the king, taking 1 etc. (of them) at a time."

Ex. 25 "O expert in (the art of preparing) mixtures, tell how many combination of mixtures can be prepared with *dhātrī*, clove, *dala*, *kesara*, *nāga*, *railā*, *vakra*, *kaṇa*, *samarica* and sugar, taking 1 etc. at a time."

PERMUTATION CORRESPONDING TO A GIVEN SERIAL NUMBER

R. 59b–60 "The number of permutations ending in a greater digit is called *Mukha*. (*Mukha*) is subtracted from the given serial number, again and again. Here, the digit corresponding to the number (i.e., the *Mukha*) which cannot be subtracted is the digit obtained. That is removed from the original order (i.e., the digits in ascending order) and the process is repeated again and again until a cipher is obtained. The remaining (digits) are (kept on the left) in the reverse (i.e. descending) order."

Explanation : Let the digits be, $d_1 < d_2 < d_3 \ldots\ldots < d_r < \ldots\ldots < d_n$ and let us find out the permutation corresponding to the serial number N. By R.55b–56 we first find out the number of permutations ending in $d_1, d_2, d_3, \ldots, d_r, \ldots, d_n$ and suppose that these are $a_1, a_2, a_3, \ldots\ldots, a_r, \ldots\ldots, a_n$. We subtract these from N, starting from a_n, in the reverse order. Suppose that $a_n, \ldots\ldots, a_{r+1}$ can be subtracted from N, leaving a remainder N_1, but a_r cannot be subtracted from N_1 (leaving a positive remainder). The d_r, the digit corresponding to a_r is the digit obtained and is written, separately. Also, d_r is removed from the digits in ascending order and again R. 55b–56 is applied to the remaining digits in ascending order and by the above process another digit is obtained. This digit is written to the left of the digit already obtained. If the number of d_r is more than 1, only one d_r is removed at a time and the process is continued with the remaining ones. The process is repeated again and again until the remainder after subtraction becomes a cipher and each time the digit obtained is written towards the left of the obtained digits. When a cipher is obtained after subtraction, this time also the corresponding digit is written to the left of the digits already obtained. Finally, the remaining digits are written in descending order (from left to right) to the left of the obtained digits. Thus the permutation corresponding to the serial number N is obtained.

Ex. 26 "In 3 previous examples, tell quickly, the 10th, the 8th and 4th permutations, separately, and from those permutations, tell quickly their serial numbers."

SERIAL NUMBER OF A GIVEN PERMUTATION

R. 61 "The numbers of permutations ending in digits greater than the last digit (as obtained) in finding (the structure of) a permutation are the measures of that. In finding the serial number (of a particular permutation), here again the learned should work (by a process) reverse to that (for finding a particular permutation corresponding to a given serial number)."

Explanation : The digits are arranged in ascending order. After that the last digit of the given permutation is considered. The sum of numbers of permutations

ending in digits greater than the last digit is found out and the last digit is removed from the given permutation and the digits in ascending order, both. The process is repeated with the digit which now occupies the last place and the sum of numbers is obtained, as above, and the last digit is again removed from the given permutation and the digits in ascending order, both. The process is repeated again and again until the digits of the given permutation become in ascending order. The sum of numbers obtained above, added to 1 (for the digits being in ascending order) gives the serial number of the given permutation.

PERMUTATION OF q DIGITS TAKING p OF THEM AT A TIME

In such permutations all digits from 1 upto q take part and in any permutation any digit may be repeated any number ($\leq p$) of times. p is called 'the Number of Places' and q, 'the Greatest Digit'.

R.62–63a "The last number of the geometric progression is the (total) number of permutations. The sequence of augmented units is multiplied by that (i.e., by the last number of the geometric progression). Except the last, the remaining (products), divided by the greatest digit, are the numbers of permutations ending in 1, 2, (etc.), in the reverse order. The sum of digits in a particular place in all permutations etc. should by obtained by the method stated earlier."

Explanation : Let the digits starting from 1 upto q take part in such permutations and the number of digits in any permutation be p, repetition of digits (any number of times $\leq p$) being allowed. Thus, here, q is the greatest digit and p, the number of places.

According to the rule,

(i) The number of permutations = q^p and

(ii) The number of permutations ending in $q, q-1, \ldots, 2, 1 = q^{p-1}$

Other results such as

(iii) the sum of digits at a particular place in all permutations,

(iv) The sum of all permutations and

(v) Total number of digits are to be obtained by R.47–48. Thus,

(vi) The sum of digits at a particular place in all permutations

$$= (1 + 2 + \ldots + q) \cdot q^{p-1}$$

$$= \frac{q(q+1)}{2} \cdot q^{p-1}$$

(vii) The sum of all permutations

$$= \frac{q(q+1)}{2} \cdot q^{(p-1)} \cdot [10^{p-1} + 10^{p-2} + \ldots\ldots + 100 + 10 + 1] \text{ and}$$

(viii) Total number of digits = $p.q^p$

Rationale : The number of permutations, for $p = 1$ will be q since each digit $\leq q$ take part in the permutations. If $p = 2$, the number of permutations will be q^2, since each of the previous permutation will be associated with each of the digit $\leq q$ (as a digit can be repeated in a permutation). In a similar manner, if $p = 3$, the number of permutations will be q^3, and so on. Thus the number of permutations = q^p.

Since digits are placed symmetrically in such permutations and the number of digits is q, so, the number of permutations ending in any particular digit [= q^p/q] = q^{p-1}.

Other results can be easily established.

Ex. 27 "Tell quickly, the number of permutations, total number of digits in all permutations and the sum of all permutations (etc.) when the greatest digit is 2 and the number of places is 6 (and when) the greatest digit is 3 and the number of places is (also) 3."

SERIAL ARRANGEMENT OF PERMUTATIONS

We have seen earlier that basic units in Sanskrit prosody are letters having a single *mātrā* and those having two *mātrās* called *laghu* and *guru*, and are represented by I and S, respectively and their role in prosody is the same as that of 1 and 2, in combinatorics. Metres in Sanskrit poetry mainly fall under the following 3 categories, (1) *Varṇa vṛttas* (2) *mātrā vṛttas* and (3) others, i.e. metres consisting of *gaṇas* (i.e., groups) of letters. *Varṇa vṛttas* are metres in which the number of letters is constant and the number of *mātrās* is arbitrary.

First of all Ācārya Piṅgala gave[33] a rule for the expansion of *varṇa vṛttas*. The rule is in the form of *sūtras* (formulae). His commentators Yādava and Halāyudha have explained the rule in detail. According to them a metre of this kind and having 1 letter has two variations S and I and the latter is written below the former : S I.

Variations of such a metre having 2 letters are obtained from those of the former by repetition with S and I, in order :

SS, IS, SI, II.

Same process of repetition of earlier variations with S and I, in order, is followed in the case of metres having more than 2 letters. Thus for a metre having 3 letters, the variations are, (1) SSS, (5) SSI

(2) ISS (6) ISI

(3) SIS (7) SII

(4) IIS (8) III.

Thus, correspondingly *prastāra* (i.e., expansion) of *varṇa vṛttas* is the same as the permutations of digits 1 and 2 (only), in which these may be repeatedly used any number of times in a permutation subject to the condition that the number of these in any permutation remains constant.

Same rule for the expansion of *varṇa vṛttas* has[34] been given by Ācārya Bharata, Jayadeva, Jaya Kīrti and Kedāra. The *Agni Purāṇa*, the *Viṣṇudharmottara Purāṇa* and the *Nārada Purāṇa* also contain[35] the same rule for such an expansion. Janāśraya, Virahāṅka, the author of *Ratna Mañjūṣa* and Hemacandra give[36] the same rule but state it differently. In mathematics, though *prastāra* is[37] in *Brāhma Sphuṭa Sidhānta* of Brahmagupta but the topic has not been translated. Ācārya Mahāvīra has[38] given the above rule for the expansion of *varṇa vṛttas*.

Nārāyaṇa has adopted the same rule for the permutations of n digits taking p at a time when a digit may be repeated in a permutation, any ($\leq p$) of times.

R. 63b–64 "Write the greatest digit in all places (in the 1st permutation). (After that) write (the next) lesser (digit) below the 1st digit and (the others) as above. Starting from the left side (each time) repeat this process (by writing) 'the digit less 1' below the 1st (available digit different from 1) and (the greatest digit in each place, left of that) until all the places in a permutation are occupied by unities. This is the process of serial arrangement (of permutations) as told by learned mathematicians."

Explanation : Let the greatest digit be q and the number of places be 1. Then the permutations are q, $(q-1)$, $(q-2)$, ..., 2, 1. If the number of places be 2, these permutations are 1st written behind q, then behind $(q-1)$ and so on, till finally behind 1. If the number of places be 3, the permutations when the number of places is 2, is 1st written behind q, then behind $(q-1)$ an so on, till finally, behind 1 and so on.

Ex. 28 "O learned, if you know, tell quickly, how the serial arrangement of permutations ending in 1, 2 (etc.) looks when the numbers of places are 6 (or) 3 and the greatest digits are 2 (or) 3, in order."

Consider the 2nd case. Here, the number of places is 3 and the greatest digit is (also) 3.

Serial Arrangement of Permutations, R. 63b–64, Ex. 28 (when $p = 3 = q$).

(1) 333	(10) 332	(19) 331
(2) 233	(11) 232	(20) 231
(3) 133	(12) 132	(21) 131
(4) 323	(13) 322	(22) 321
(5) 223	(14) 222	(23) 221
(6) 123	(15) 122	(24) 121
(7) 313	(16) 312	(25) 311
(8) 213	(17) 212	(26) 211
(9) 113	(18) 112	(27) 111

NUMBER OF PERMUTATIONS IN WHICH A DIGIT OCCURS r TIMES. *LAGAKRIYĀ*, BINOMIAL COEFFICIENT $C(p, r)$ AND SEQUENCE OF BINOMIAL COEFFICIENTS

We have seen earlier that 6 *pratyayas* regarding Sanskrit metres were very important for the knowledge and development of Sanskrit prosody, *Lagakriyā* being one of them.

The process for finding the number of variations of a *varṇa vṛtta* having n letters which have a definite number of letters of a single *mātrā* (or of 2 *mātrās*) has been named *Lagakriyā* or *Galakriyā* or *Laghukriyā* etc. by authorities on metrical sciences.

We have stated earlier that the expansion of *varṇa vṛttas* having n letters, in prosody, is the same as the permutations of 1, and 2 when these can be repeated any number of times in a permutation such that the number of places is n, in combinatorics. In this light *lagakriyā* means to find $C(n, r)$, $r = 0, 1, 2, \ldots, n$ and thus originated the rules for the calculation of binomial coefficients and of the sequence of binomial coefficients and studies of their properties, in India. These mostly given in the form of figures of numbers have been named differently by different authors. Moreover, they have given different rules for them, as well.

Though there is a long list of teachers of Sanskrit prosody prior to Piṅgala, his *Chanaḥ Sūtra* is the earliest extant work which contains a rule[39] for the same. His figure of numbers has been named *Prāsāda-prastāra*[40] and *Meru-prastāra*[41] by his

commentators Yādava and Halāyudha, respectively and is based on the formula, $C(n, r-1) + C(n, r) = C(n+1, r)$. The *Agni Purāṇa* mentions *Meru-prastāra* and states Piṅgala's rule for its formation. The rule has also been given[43] by Virahāṅka, Trivikrama and the author of *Prākṛta Paiṅgala*. Moreover, Gopāla and Trivikrama have illustrated the rules with figures of *meru-prastāra*. *Lagakriyā* has[44] been described by Ācārya Bharata in his *Nāṭya Śāstra*.

Meru has[45] been mentioned by Brahmagupta but his rule has not been deciphered so far. Another form of figure of numbers called *Sūcī-Prastāra* also based on the formula $C(n, r-1) + C(n, r) = C(n+1, r)$ and containing sequence of binomial coefficients for *lagakriyā* has been[46] given by Varāhamihira (510 A.D.), Janāśraya, Virahāṅka, Jayadeva and the author of *Ratna Mañjūṣā*, Jayakīrti, Kedāra, Trivikrama and Hemacandra. Bhaṭṭotpala (966 A.D.) has[47] explained Varāhamihira's rule in detail.

Another form of figure of numbers called *Khaṇḍa-meru*, and resembling to a great extent to *sūci-prastāra* in its formation, alongwith the figure of numbers[48] has been given by the author (commentator?) of the *Ratna Mañjūṣā* and Punnāgacandra has been credited with its discovery. Bhāskara II mentions[49] *Khaṇḍa-meru* but has not described it. Still another form of figure of numbers called *Loṣṭa Prastāra* has[50] been given by Varāhamihira (510 A.D.). Bhaṭṭotpala (966 A.D.) has[51] explained his rule in detail. We have seen earlier (R. 58) that still another rule to find the value of $C(n, r)$, i.e.,

$$C(n, r) = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}, \; r = 1, 2, 3, \ldots\ldots, n$$

has been given by Mahāvīra, Śrīdhara, Bhāskara II and Nārāyaṇa Paṇḍita and the same rule, given by an ealier authority, has been quoted by Bhaṭṭotpala (966 A.D.).

Here it will not be out of place to take a very brief look at another property of binomial coefficients known to the Indians from very early times, viz.,

$$\sum_{r=0}^{n} C(n, r) = 2^n \ldots\ldots (1)$$

We have seen in the very beginning that *Caraka Saṁhitā and Suśrutu Saṁhitā* find the number of various kinds of mixtures which can be prepared from 6 basic tastes which is equivalent to find the values of $C(6, r)$, $r = 1, 2, \ldots\ldots, 6$ and then find that $\sum_{r=1}^{6} C(6, r) = 63$.

Piṅgala gave[52] a rule to find the number of variations of a *varṇa vṛtta* having n letters ($= N$, say) as 2^n. His rule for *lagakriyā*, i.e., to find $C(n, r)$, $r = 0, 1, 2, \ldots, n$, stated above, shows that he was well aware of formula (1) given above. Moreover, Ācārya Bharata, Jayadeva, Jayakīrti and Kedāra etc. have[53] given the value of N in both the forms, which are equivalent to

$$N = \sum_{r=0}^{n} C(n, r) \text{ and } N = 2^n .$$

Nārāyaṇa's rule given below is an extension of *lagakriyā* for *varṇa vṛttas* to (such) permutations of digits.

R. 65–66 "Taking 'the greatest digit less 1' as the common ratio, write the geometric progression in the reverse order and 'the sequence of polynomial coefficients when the greatest digit is 2', below that. Multiply (the numbers of) the sequence of polynomial coefficients by the corresponding numbers above these, severally. These (i.e., the products) happen to be the number of permutations without 1, containing 1, once, twice, thrice etc. or without 2, containing 3, once, twice, thrice etc."

Explanation : By R.17b–18a, the geometric progression will be $1, (q-1), (q-1)^2, \ldots\ldots, (q-1)^{p-1}, (q-1)^p$, since the common ratio of the geometric progression is $(q-1)$ by this rule. Written in the reverse order, the progression will be,

$$(q-1)^p, (q-1)^{p-1}, \ldots\ldots, (q-1)^2, (q-1), 1.$$

So, its $(r+1)$th term will be $(q-1)^{p-r}$. Also, for $q = 2$,

$$u(p, q, r) = C(p, r), r = 0, 1, 2, \ldots\ldots, p.$$

So, $(r+1)$th term of this sequence will be $C(p, r)$. According to the rule, number of permutations in which a digit occurs r times

$$= C(p, r)(q-1)^{p-r}, r = 0, 1, 2, \ldots\ldots, p.$$

Rationale : Let p be the number of places and q, the greatest digit, so that all the digits between 1 and q take part in the permutations. Consider any one of the permutations in which a particular digit occupies r particular places. Since one digit has occupied r places, the rest of $(p-r)$ places will be occupied by $(q-1)$ digits and this can be done in $(q-1)^{p-r}$ ways (R. 62) because r places are occupied by that particular digit. Now, consider the number of permutations in which a particular digit occupies any r places. This can be done in $C(p, r)$ ways as the number of places is p. Therefore, the number of permutations in which a digit occurs r times

$$= C(p, r).(q-1)^{p-r} \text{ and hence the rule.}$$

Ex. 29 "If (you) know, tell the number of permutations without 1, containing 1 once, twice, thrice, etc., in the two (previous) examples."

This example refers to 2 problems given in Ex. 28. According to the rule, the desired numbers of permutations in the 1st example will be 1, 6, 15, 20, 15, 6, 1 and those in the 2nd example will be 8, 12, 6 and 1, in order.

ALTERNATIVE RULE

R. 67a "Numbers of the sequence in rows of the *meru* are the measures of permutations without 1, contain 1 once, twice etc."

Let p and q have their usual meanings. According to the rule, numbers in cells of $(p + 1)$th row of the (*su*)*meru* gives the numbers of permutations without 1, containing 1 once, twice etc. or without 2, containing 2 once, twice etc. etc. It is so due to R. 65–66 because, $(p + 1)$th row of the (*su*)*meru* contains the numbers

$$C(p, r).(q-1)^{p-r},\ r = 0, 1, 2, \text{etc.}$$

THE BINOMIAL THEOREM

R. 67b "Sums of numbers of the sequences in rows of the *sumeru* happen to be in geometric progression."

From the law of formation of the *sumeru*, we have seen (R. 36b–39) that $(r + 1)$th row of the *sumeru* contains the expressions,

$$C(r, 0)s^r, C(r, 1)s^{r-1}, C(r, 2)s^{r-2}, \ldots, C(r, r-1)s, C(r, r), \text{ where } s = q - 1.$$

(See Fig. 3).

Now, according to this rule sums of numbers in successive rows of the *sumeru* form a geometric progression. Obviously, sums of numbers in $(r + 1)$th rows, $r = 0, 1, 2, 3, \ldots, r$, are

$$1, C(1, 0)s + C(1, 1), C(2, 0)s^2 + C(2, 1)s + C(2, 2),$$

$C(3, 0)s^3 + C(3, 1)s^2 + C(3, 2)s + C(3, 3), \ldots$, in order and clearly, these are in geometric progression

$$(1 + s)^r,\ r = 0, 1, 2, 3, \ldots,$$

whose 1st term is 1 and whose common ratio is $(1 + s)$. Thus, according to the rule,

$$C(r, 0)s^r + C(r, 1)s^{r-1} + C(r, 2)s^{r-2} + \ldots + C(r, r-1)s + C(r, r)$$
$$= (1 + s)^r,\ r = 0, 1, 2, 3, \ldots$$

and this is a form of the binomial theorem stated inversely for $r = 0, 1, 2, 3, \ldots$. The theorem has been described for an integral index, where s is also integral.

Origin and development of the studies of the binomial coefficients and their properties in India has been described earlier. In this connection it is needless to say that the binomial coefficients along with the relations

$$C(n, r) + C(n, r - 1) = C(n + 1, r)$$

$$\text{and } C(n, 0) + C(n, 1) + C(n, 2) + \ldots\ldots + C(n, n) = 2^n,$$

were known to the Indians, at least, since the time of Piṅgala (about 200 B.C. or earlier), culminating in the description of the binomial theorem by Nārāyaṇa Paṇḍita (1356 A.D.) in the above form.

At-Karkhi (died c. 1029 A.D.) has used[54] a slightly old fashioned form of mathematical induction to demonstrate the expansion of $(a + b)^3$ and $(a + b)^4$. Al-Samawal (d. 1175 A.D.) reports[55] in his *al-Bāhir* that the table of binomial coefficients, its formation law, i.e.,

$$C(n, r) = C(n - 1, r) + C(n - 1, r - 1)$$

and the expansion

$$(a + b)^n = C(n, r)a^{n-r}b^r$$

for an integer n can be found in a text by al-Karkhi. 'Umar Khayyām (c. 1038/48–1123/24) claimed[56] to have discovered the law. Al-Zanjānī uses the binomial theorem, inversely[57], to extract the square-root of a polynomial and gives the expansion upto $n = 7$. Al-Tūsī has[58] displayed the socalled Pascal's triangle in his book '*Handbook of Arithmetic using Board and Dust*' (1265 A.D.) and explained its construction in it. He uses the triangle in the extraction of roots. Discussion of the theorem also finds[59] place in the writings of Ibn-al-Bannā (died 1321 A.D.) and Ibn Zurayq (14th century). Al-Kāshi, an assistant of Ulugh Beg (1393–1449 A.D.) wrote[60] a short treatise on arithmetic and geometry in Persian which contains reference to the arithmetical triangle.

In China the (socalled) Pascal's triangle was probably, first described[61] by Liu Ju-Hsieh, a contemporary of Chia Hsien (c. 1100 A.D.), in his book *Fu Chi Shih So*. Chia Hsien expounded the triangle in 1100 A.D. His method was called Li-Chhēng Shih So. The earliest extant Chinese representation of the triangle is[62] found in Yang Hui's *Hsiang Chich Chiu Chang Suan Fa* of 1261 A.D. Chu Shi Kie (Chu Shih Chieh) gives[63] the (so called) Pascal's triangle in his *Sṣu Yuan Yu Chien* of 1303 A.D.

In Europe, the socalled Pascal's triangle appeared[64] on the title page of *Arithmetic* of Apinius (1527 A.D.). Stifel gave[65] the triangle in the first part of his *Arithmetica Integra* (1544 A.D.). The triangle also appeared in Scheubel's *De Numberis et Diversis Rationibus* (1545 A.D.). Later on, the triangle appeared[66] in the writings of Peletier, Trenchant (1566 A.D.) and Tartagalia (1556 A.D.).

Bombelli (1572 A.D.) gave[67] the triangle for $n = 7$ and Oughtred (1631 A.D.), for $n = 10$.

B. Pascal gave[68] the triangle in his *Traité du triangle arithmetique*, written in 1653 A.D. and published posthumously in 1665 A.D. In it he set forth numerous discoveries related to this array. He used the triangle partly for finding the binomial coefficients and partly for finding the value of $C(n, r)$. Newton gave[69] the binomial theorem for negative and fractional indices in his letters to Oldenburg in 1676 A.D. Leibnitz (1695 A.D.), J. Bernoulli and De Moivres generalized[70] the binomial theorem into the polynomial theorem.

Ex. 30 "How many are the numbers of permutations without 1, containing 1 once etc., without 2 etc., without 3 etc., in the 2 previous examples?."

The example given here refers to 2 problems given in Ex. 28. Nārāyaṇa has given solutions of the problems with the help of two figures of numbers. One figure of numbers is the same as that given after Ex. 12. The other is as given below.

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Fig. 7. (R. 67a, Ex. 30)

NUMBER OF PERMUTATIONS HAVING A GIVEN SUM

R.68 "Consistent with the greatest digit and the number of places, write the sequence of polynomial coefficients below the *yaugika* sequence. Here, that i.e., a number of the sequence of polynomial coefficients is the number of permutations whose sum is determined by a number in the upper place."

Let p, q and $u(p, q, r)$ have their usual meanings so that the *yaugika* sequence will be $pq, pq - 1, \ldots, p$ (R. 22b–23a). According to the rule, the number of permutations whose sum of digits are $pq, pq - 1, pq - 2, \ldots pq - r, \ldots, p$ will be given by $u(p, q, r), r = 0, 1, 2, \ldots, r, \ldots, p(q - 1)$.

The rule immediately follows from R. 67a.

Ex. 31 "O learned if you know, 'the concatenation of numbers' tell the numbers of permutations, (whose) sums are equal to the numbers of the *yaugika* sequence, in the two (previous) examples."

The examples referred to are Ex. 28 (both cases). In the 1st example ($p = 6$, $q = 2$), the numbers of permutations whose sums are 12, 11, ..., 6 will be 1, 6, 15, 20, ..., 1 and in the 2nd example ($p = 3$, $q = 3$), the number of permutations whose sums are 9, 8, ..., 3 will be 1, 3, 6, 7, ..., 1, in order.

NUMBER OF PERMUTATIONS HAVING A GIVEN SUM AND NUMBER OF SUCH PERMUTATIONS ENDING IN A PARTICULAR DIGIT

R. 69 "Sum of numbers of 'the partial sequence of polynomial coefficients' is definitely the number of permutations. Those number of 'the partial sequence of polynomial coefficients' are the numbers of (such) permutations ending in 1 etc."

Let p, q and $u(p, q, r)$ have their usual meanings. Also, let the sum of digits of a particular permutation be n and let the rth term of the partial sequence of polynomial coefficients be $V(p, q, n, r)$, [R. 23b–25a]. According to latter part of the rule, number of such permutations ending in

$$r = V(p, q, n, r)$$

$$= u(p-1, q, (p-1)\,q + p - n + r)$$

$$= \text{coefficient of } x^{(p-1)q+p-n+r} \text{ in the expansion of}$$

$$(1 + x + x^2 \ldots\ldots + x^{q-1})^{p-1}, \text{(R.23b–25a)}$$

According to former part of the rule, total number of such permutations

$$= \sum_{n=p}^{pq} \sum_{r=1}^{q} V(p, q, n, r)$$

$$= \sum_{n=p}^{pq} \sum_{r=1}^{q} u(p-1, q, (p-1)\,q + p - n + r), \text{(R. 23b–25a)}$$

Rationale : In such type of permutations, total number of permutations is q^p (R. 62a) and the number of permutations having a given sum is obtained from $u(p, q, r)$, $r = 0, 1, 2, \ldots\ldots, q$. Now, the number of permutations ending in a particular digit is q^{p-1} (R.62b) and so the number of permutations having a given sum in these permutations will be obtained from $u(p-1, q, r)$, $r = 0, 1, 2, \ldots\ldots, q$ and hence the latter part of the rule. Former part of the rule is obvious.

Ex. 32 "O learned, tell the (total) number of permutations (and) the number of permutations ending in 1 etc., when the greatest digit is 3, the number of places is (also) 3 and the sum of digits in (each of such) permutations is 6."

SERIAL NUMBERS OF PERMUTATIONS HAVING A GIVEN SUM

R. 70 "The serial numbers of permutations (whose) sum (of digits) are in the *yaugika* sequence should be obtained from the *patākā* and the numbers in the cells of the column below that (i.e., below the sum) are (such) serial numbers of permutations, in order."

Explanation : Let p and q have their usual meanings. Formation of *patākā* and insertion of numbers in that for given p and q have been dealt with earlier (R. 29–36a). Numbers of the corresponding *yaugika* sequence is written (horizontally) above that, in order. A number of the *yaugika* sequence in a cell is the sum of digits of permutations whose serial numbers are given by numbers in cells of the column containing the sum, in order. Earlier, the author of *Ratna Mañjūṣā* has given[71] rules for its formation and its application (See R. 29–36a).

Ex. 33 "O learned, if you know, tell me separately, the serial numbers of permutations in the expansion when the greatest digit is 3, the number of places is (also) 3 (and) the sum of digits are 9, 8, 7, 6, 5, 4 and 3."

Here $p = 3 = q$. The corresponding *yaugika* sequence will be 9, 8, 7, ..., 3. Writing these (horizontally) above the *patākā* (Fig. 2, R.29–36a), we observe that serial numbers of permutations whose sums are 9; 8; 7; . . . . .; 3 are 1; 2, 4, 10; 3, 5, 7, 11, 13, 19; . . . . .; 27, in order.

PERMUTATION CORRESPONDING TO A GIVEN SERIAL NUMBER

R.71–72a "The (smallest) digit which when added to the given serial number makes the sum exactly divisible by 'the greatest digit' is 'the digit obtained'. Also, repeat this process with the (successive) quotients. (Each of) the additives (i.e., the digits obtained) increased by 1, (kept) in order, is the permutation corresponding to the (given) serial number."

Let p and q have their usual meanings, N, the given serial number and d', the smallest digit such that $N + d'$ be divisible by q without remainder. Suppose, the quotient is N'. Again, let $N' + d''$ be divisible by q without remainder and so on. Let $d' + 1 = d_1, d'' + 1 = d_2, d''' + 1 = d_3, \ldots\ldots, d'''^{\cdots\cdots} + 1 = d_p$. Then according to the rule, $d_1 d_2 \ldots\ldots, d_p$ is the Nth permutation.

Rationale : Consider $(\ldots\ldots + mq^2 + nq + r)$th permutation, where m, n, r are each $< q$. Division of $(\ldots\ldots + mq^2 + nq + r)$ by q will give the remainder r. Let $r + r_1 = q$ so that, $r_1 = q - r$, is the additive. If we add 1 to this, we get $q - r + 1$. But $(\ldots\ldots + mq^2 + nq + r)$th permutation will have $(q - r + 1)$ as its 1st digit as the 1st permutation has q as its 1st digit and after each permutation, the 1st

digit becomes 1 less than the previous one and the same situation arises after each q permutations. Thus we get the 1st digit of the permutation as stated by the rule. The 2nd, 3rd etc. digits of the permutation change after q, q^2, etc. permutations in the same manner in which the 1st digit changes. Since for each next digit we proceed with the successive quotients so what is true for the 1st digit, will be true for the 2nd, 3rd etc. digits, as well and hence the rule.

The rule is[72] an extension of the rule given by Ācārya Piṅgala, Ācārya Bharata, the author of the *Agni Purāṇa*, Janāśraya, Virahāṅka, Jayadeva, the author of Ratna Mañjūṣā, Jayakīrti, Kedāra and other authorities on Sanskrit prosody for finding the variation corresponding to a given serial number in the serial arrangement of all variations of the metre. Brahmagupta has[73] described *naṣṭa* but his rule has not been deciphered so far. Mahāvīra's rule for *naṣṭa* is[74] the same as that given by authorities on Sanskrit prosody.

Ex. 34 "O friend, where the greatest digit is 3 and the number of places is (also) 3, tell how the 20th permutation looks."

Solution : Here, $p = 3 = q$. The (smallest) digit which when added to the given serial number, 20 makes the sum exactly divisible by the greatest digit 3 is 1, so 1 is set apart. Also, the quotient after division is 7, the process is followed with 7 and we get 2 as the additive and 3 as the quotient and 2 is set apart. Again the process is followed with 3. We get 0, which is set apart. Since $q = 3$ and 3 additives have been obtained, adding 1 to each of the digits set apart, we get 231 which is the desired 20th permutation.

Serial Number Corresponding To A Given Permutation

R.72b–74 "(Write) the digits of the given permutation, separately, after subtracting unity (from each of them) and (also), 'the greatest digit' below (each of them). There, starting from the end, in the reverse order, subtract the digit lying above from 'the greatest digit'. Multiply (the difference) by that beside it and (lying below). Again, subtract the digit lying above (from the product) and in this way, repeat the process with all the terms. This (i.e., the result) is the serial number of the given permutation as enumerated by learned mathematicians."

The serial number of a given permutation is obtained by a process reverse to that meant for finding the permutation when its serial number is given. Let p and q have their usual meanings and let the given permutation be $d_1 d_2 \ldots\ldots d_{p-1} d_p$. According to the rule, the given serial number

$$= q\,(\ldots\ldots[q\{q-(d_p-1)\}-(d_{p-1}-1]\ldots\ldots)-(d_1-1)\,.$$

The rule is[75] an extension of the rule given by Ācārya Piṅgala, the author of the *Agni Purāṇa*, Janāśraya, Virahāṅka, the author of *Ratna Mañjūṣā* and other authorities on Sanskrit prosody for finding the serial number of a given variation of a metre amongst serial arrangement of all variations of the metre. Brahmagupta has[76] described *uddiṣṭa* but his rule has not been deciphered so far. Mahāvīra has[77] given a slightly different rule related to *varṇa vṛtta* which has also been given by Jayadeva, Jayakīrti and other authorities on Sanskrit prosody.

Ex. 35 "O best among learned, if you know the process of liberation from the bond of concatenation of numbers, tell quickly, the serial number of permutation 231 obtained from the expansion (of digits) where the greatest digit (and the number of places, each) is 3."

Solution : Subtracting 1 from each digit of the given permutation, 1, 2, 0 are obtained which are written separately and the greatest digit, 3, is written below each of them. Thus, we get,

$$\begin{matrix} 1 & 2 & 0 \\ 3 & 3 & 3 \end{matrix}$$

Starting from the end, the process is effected leftwards. Subtracting 0, the digit lying above, from 3, the digit lying below, 3 is obtained. Multiplying this 3, by the digit towards the left and lying below, i.e., 3, we obtain 9. Again, the next digit lying above (and towards the left, i.e., 2) is subtracted from the product 9 and thus 7 is obtained. Multiplying 7 by 3, the digit towards the left and lying below, we get, 21. Subtracting 1, the digit lying above from 21, we obtain 20, which is the desired serial number of the given permutation amidst serial arrangements of all permutations with the given digits.

PARTITIONS OF A NUMBER AND FIBONACCI NUMBERS

A partition of the positive integer n is[78] a set of positive integers whose sum is n. If the set has p numbers, it is a p–partition of n. Partition dealt with by Bhāskara II and Nārāyaṇa slightly differs from partition used in modern mathematics. Moreover, to understand the concepts and ideas of authorities on Sanskrit metrical sciences regarding partitions while dealing with Sanskrit prosody a slightly different notation has been used for such partitions, throughout this paper.

A partition of the positive integer n into a set of p–ordered digits has been indicated by partition $(0d_i)$, $i = 1, 2, \ldots, q$ [or p–partition $(0d_i)$], $i = 1, 2, \ldots, q$. Thus, in modern mathematics, (1, 2, 1, 3) and (1, 1, 2, 3) are the same partition (or 4-partition) of 7, while these are two different partitions $(0d_i)$, $i = 1, 2, 3$ [or 4-partitions $(0d_i)$, $i = 1, 2, 3$] of 7.

Nārāyaṇa's partitions are partitions $(0d_i)$, $i = 1, 2, \ldots\ldots, q \leq 9$. If the set has p digits it is Nārāyaṇa's p-partition $(0d_i)$, $i = 1, 2, \ldots\ldots, q \leq 9$. Hereafter, until otherwise stated, partition means partition $(0d_i)$, $i = 1, 2, \ldots\ldots, q \leq 9$. Rules regarding such partitions have been given later on in R. 75, onwards.

The name of L. Fibonacci is attached with the numbers 0, 1, 1, 2, 3, 5, 8, 13, ... in which $u_n = u_{n-1} + u_{n-2}$, where u_n denotes the nth term of the sequence. But the sequence was well known in India prior to him. Indian authorities on metrical sciences have dealt with the sequence in their work on prosody.

We have seen earlier that metres in Sanskrit poetry mainly fall under the categories (1) *varṇa vṛtta* (2) *mātrā vṛtta* and (3) others. A *mātrā vṛtta* is a metre in which the number of *mātrās* is constant and the number of letters is arbitrary.

The process for expansion (*prastāra*) of a *mātrā vṛtta* is similar to that of a *varṇa vṛtta*. Thus the expansion of a *mātrā vṛtta* having *n mātrās* is obtained by writing all the variations of a *mātrā vṛtta* having $(n-2)$ *mātrās* behind S and then those having $(n-1)$ *mātrās* behind I, in order. Since a *mātrā vṛtta* having a single *mātrā* has only one variation, viz., I and that having 2 *mātrās* has 2 variations, viz., S and II, those having 3 or 4 or 5 *mātrās* have 3 or 5 or 8, variations, in order, viz.,

IS,	or	SS	or	ISS,	in order
SI		IIS		SIS	
III		ISI		IIIS	
		SII		SSI	
		IIII		IISI	
				ISII	
				SIII	
				IIIII	

Here, it is easily seen that the numbers of variations of *mātrā vṛttas* form the sequence of numbers which are the socalled Fibonacci numbers. For, the numbers of variations of metres having 1, 2, 3, 4, 5, 6, ... *mātrās* are 1, 2 ,3, 5, 8, 13, ..., respectively, as seen above and the latter numbers are the socalled Fibonacci numbers. Let t_n denote the number of variations of the metre having *n mātrās*. Then from the process of expansion of *mātrā vṛttas* as described above, it is clear that

the numbers of variations of *mātrā vṛttas* obey the law, $t_n = t_{n-1} + t_{n-2}$, which is also the law for the formation of the socalled Fibonacci numbers. Thus originated the concept of the socalled Fibonacci numbers and studies of their properties in India.

It is easily seen that the variations of a *mātrā vṛtta* ḥaving n matras are correspondingly, partitions $(0d_i)$, $i = 1, 2$, of n. Thus originated the studies of partitions (of a number) and its properties, in India. Hence, it can be safely concluded that the concept of Fibonacci numbers and partitions and studies of their properties in this country is at least as old as the origin of Sanskrit and Prakrit prosody, here in. Here, we also note that Nārāyaṇa's partition is an extension of the former partition for in the former case, $i = 1, 2$, only, whereas in Nārāyaṇa's partition, $i = 1, 2, \ldots\ldots q \leq 9$.

Ācārya Piṅgala is the first authority on metrical science whose writings indicate a knowledge of the socalled Fibonacci numbers. According to his commentator Yādāva, his rule which may be translated as "and the two mixed" is also[79] meant for the expansion of *mātrā vṛttas*. According to his commentators Yādāva and Halāyudha, Piṅgala's rule for the variations of a *gaṇa* (i.e., a group of letters having 4 *mātrās*) in an *āryā* metre states[80] the variations in the order in which they will be obtained by this rule.

Ācāraya Bharata specifically mentions[81] that the rule "and the two mixed" is also meant for *mātrā vṛttas*. The *Viṣṇudharmottara Purāṇa* is[82] the next work, whose chapter on Sanskrit prosody after giving the rule for the expansion of *varṇa vṛttas* mentions that the same is the rule for the expansion of *mātrā vṛttas*.

Ācārya Virahāṅka is[83] the first authority on metrical science who gives rules for the formation of numbers of variations of *mātrā vṛttas* in specific words and his commentator Gopāla explaining the rule in detail, specifically mentions[84] the numbers of variations of *mātrā vṛttas* having 1, 2, 3, 4, 5, 6, 7, ... *mātrās* as 1, 2, 3, 5, 8, 13, 21, ... (the socalled) Fibonacci numbers, prior to Avant L. Fibonacci.

Ācārya Hemacandra is the next authority on prosody who gives[85] the rule for the formation of numbers of variations of *mātrā vṛttas* which is the same as the rule for the formation of Fibonacci numbers, in his *Chandonuśāsana* (1142 A.D.) and specifically mentions the numbers 1, 2, 3, 5, 8, 13, 21, 34, in this regard also, prior to A.L. Fibonacci.

The author of *Kavidarpaṇa* (13th century), another work on prosody, also gives[86] the rule. *Prākṛta Paiṅgala* contains[87] several rules regarding the socalled Fibonacci numbers. These will be dealt with later on at proper places.

TOTAL NUMBER OF PARTITIONS AND NUMBER OF PARTITIONS ENDING IN A PARTICULAR DIGIT

R. 75 "Last number of the *sāmāsikā* sequence is the number of partitions[88]. Other numbers (of the sequence) in the reverse order (i.e., from right to left), starting from the penultimate, are the numbers of partitions ending in 1 etc. (in order)." Let t_n denote the nth term of the *sāmāsikā* sequence. According to the rule, the number of partitions of $n = t_{n+1}$ and those of partitions ending in 1, 2, ..., n are $t_n, t_{n-1}, \ldots, t_1$, in order."

Rationale: By actual observation we see that both parts of the rule are true for $n = 1$, 2 or 3. Thus for $n = 3$, partitions of $n = t_{n+1}$ and those, ending in 1, 2, and 3 are t_3, t_2 and t_1, i.e., t_1, t_2 and t_3 in the reverse order. Now for any $q = n \leq 9$, the partitions of n are obtained by writing n and then partitions of $n-1, n-2, \ldots, 2, 1$, behind $1, 2, \ldots, n-1$, which are $t_1, t_2, \ldots, t_{n-1}, t_n$ i.e., $t_n, t_{n-1}, \ldots, t_2, t_1$ in the reverse order. This proves the latter part of the rule and since, in a *sāmāsikā* sequence $t_{n+1} = t_1 + t_2 + \ldots + t_n$, this proves the former part of the rule.

When q < n, for any n, partitions involving digits greater than q and ending in $n, n-1, \ldots, q+1$ are $t_1, t_2, \ldots, t_{n-q}$ and are omitted from the partitions and so we do not get partitions ending in $n, n-1, \ldots, q+1$. Obviously, then partitions ending in $q, q-1, \ldots, 1$ are given by $t_{n-q+1}, t_{n-q+2} \ldots t_n$. Also, total number of partitions $= t_{n-q+1} + t_{n-q+2} + \ldots + t_n$. But in this case, in a *sāmāsikā* sequence $t_{n+1} = t_{n-q+1} + t_{n-q+2} + \ldots t_n$.

Therefore, both parts of the rule are true in this case also. Hence the rule.

We have seen above that variations of a *mātrā vṛtta* having *n mātrās* are given by (n + 1)th number in the socalled Fibonacci numbers, 1, 1, 2, 3, ... From the rule of expansion of *mātrā vṛttas* it is also clear that the numbers of variations of such a *mātrā vṛtta* ending in I (a *laghu*) and S(a *guru*) will be given by nth and (n –1)th socalled Fibonacci numbers. Moreover, such variations of *mātrā vṛttas* are correspondingly partitions $(0d_i)$, $i = 1, 2$ of n. We have also seen that Nārāyaṇa's partition is an extension of partition $(0d_i)$ $i = 1, 2$ and that Nārāyaṇa's *sāmāsikā* sequence is a generalization of the socalled Fibonacci numbers (R. 13–15a).

Obviously, then for a particular q the number of Nārāyaṇa's partitions of n and numbers of partitions of n, ending in 1, 2, 3, ..., $n-1$, n, should be given by $(n+1)$th and nth, $(n-1)$th, $(n-2)$th . . . . . 2nd and 1st terms of the corresponding *sāmāsikā* sequence (for that value of q), in order. The rule describes the same fact.

Thus, Nārāyaṇa's rule is an extension of the rule given by earlier authorities on Sanskrit prosody for number of variations of *mātrā vṛttas* and those of variations of such *vṛttas* ending in I an S.

Ex. 36–37 "O friend, if you know (how) to liberate (yourself) from the knot produced from the concatenation of numbers by many groups of rules, tell the number of partitions, the number of partitions ending in 1, 2, 3, etc., the measures of the numbers of digits 1, 2, 3, etc. produced from them, separately, the numbers of partitions occupying 1, 2, 3, 4 etc. places (i.e., the number of p–partitions, $p = 1, 2, 3, 4$ etc.), their sums (and the total sum) where the sum of digits is 7 (and the greatest digit is (also) 7 and (also) tell these, when the greatest digit is 3."

Consider the 2nd case. Here, the greatest digit = 3 and the sum (of digits, i.e., the number to be partitioned) = 7. So, the corresponding *sāmāsikā* sequence will be 1, 1, 2, 4, 7, 13, 24 and 44. (R. 13–15a, Ex. 3). Therefore, the total number of partitions = 44. Also, writing other numbers of the *sāmāsikā* sequence in the reverse order (i.e., from right to left), starting from the penultimate and taking only 3 of these (as 3 is the greatest digit), we find that 24, 13 and 7 are the numbers of partitions of 7, ending in 1, 2 and 3, in order. Rule for the expansion of partitions of a number has been dealt with later on, in R. 79–80.

NUMBER OF A PARTICULAR DIGIT AND NUMBER OF ALL DIGITS IN PARTITIONS OF A NUMBER

R. 76. "Numbers of the *pātāla* sequence, produced from the same (sum and the same) greatest digit, in the reverse order (starting from the last) are the numbers of 1 etc. (and) their sum is the number of digits in all partitions (of the number)."

Let T_n and t_n denote the nth terms of the *pātāla* sequence and *sāmāsikā* sequence, respectively. According to the rule the numbers of digits 1, 2, 3, ... q in the partitions of n are $T_{n+1}, T_n, T_{n-1}, \ldots\ldots, T_{n-q+2}$ and the number of all the digits in such partitions = $T_{n+1} + T_n + T_{n-1} + \ldots + T_{n-q+2}$.

Rationale : From actual observation we find that the rule is true for $n = 1, 2$ or 3, if $q = 2$ or 3. Now let $q = n \leq 9$. Then $T_{(n+1)} = \sum_{r=1}^{n} T_r + t_n$.

Also, from the serial arrangement of partitions, given later on (R. 79–80) it is clear that the number of unities in the partitions of n will be equal to those of unities

present in all the previous partitions added to the number of partitions of $(n-1)$,

i.e., $\sum_{r=1}^{n} T_r + t_n = T_{n+1}$.

So, this rule is true for the number of unities in the partitions of n. Similarly, the number of unities in the partitions of partitions of $n-1, n-2, \ldots\ldots, 1$ will be $T_n, T_{n-1}, \ldots\ldots, T_2$, in order.

But from the serial arrangement (R. 79–80) we also observe that numbers of 2, 3, 4, ..., etc. in the partitions of n will be the same as those of 1, 2, 3, ..., etc. in the partitions of $(n-1)$, in order and similarly, the numbers of 2, 3, 4, ... etc. in the partitions of $(n-1)$ will be the same as those of 1, 2, 3, ..., etc., respectively, in the partitions of $(n-2)$ and so on and so forth. Therefore, the numbers of 2, 3, 4, ..., etc. in the partitions of n will be equal to $T_n, T_{n-1}, T_{n-2}, \ldots\ldots$, etc. in order. Thus, the rule is true in this case. The above reasoning can be, similarly, modified to deal with case when $q < n$ and n has any integral value whatsoever. Therefore, the 1st part of the rule is true in all cases. The 2nd part of the rule is obvious. Hence, the rule.

The author of *Prākṛta Paiṅgaḷa* gives[89] a rule to find the numbers of *gurus* and *laghus* (and so of all the letters) amongst all variations of a *mātrā vṛtta* having *n mātrās*. Correspondingly, the rule is to find the numbers of the digits 1 and 2 (and so of all the digits) in the partitions $(0d_i)$, $i = 1, 2$, of n. Nārāyaṇa's rule is an extension of the same for numbers of digits 1, 2, ... , q in partitions of n.

Consider the 2nd case of the previous example (Ex. 36–37). Let q and n have their usual meanings. Here, $n = 7$ and $q = 3$. So, the *sāmāsikā* sequence will be 1, 1, 2, 4, 7, 13, 24 and 44 (R. 13–14, Ex. 3) and the corresponding *pātāla* sequence will be 0, 1, 2, 5, 12, 26, 56 and 118 (R. 15–17, Ex. 4). Since the greatest digit = 3, 3 numbers of the *pātāla* sequence, in the reverse order, starting from the last are, 118, 56 and 26. According to the rule, 118, 56 and 26 are the numbers of digits 1, 2 and 3, in order and their sum, i.e., 200 is the total number of digits in all partitions of 7. Care has to be taken that no digit greater than 3 is used in these partitions.

NUMBER OF p–PARTITIONS, THEIR SUM AND SUM OF ALL PARTITIONS, OF A NUMBER

R. 77–78 "Numbers in cells of the last column of the (corresponding) *matsya-meru* happen to be the numbers (of p-partitions of the given number)[90]. Multiply them by the sum (i.e., the given number, separately, and then) divide (the products) by their respective number of places. Multiply the quotients by (their respective) sequence of unseparated units. Their sum (i.e., the sum of the products) definitely happens to be the sum (of all partitions of the given number)."

Explanation : While finding the number of p–partitions of the given number, one must start from such value of p as will make the number in the lowest cell of the last column correspond to the value of p given by the number. Let n, p and q have their usual meanings. Obviously, p will take all integral values such that $(n/q) \leq p = n$. According to former part of the rule, the number of p-partitions of n

$= u(n-k, q, k), \quad k = 0, 1, 2, \ldots\ldots$, in order

= the coefficient of x^K in the expansion of

$(1 + x + x^2 + \ldots\ldots + x^{q-1})^{n-k}$.

According to latter part of the rule, the sum of all partitions

$= \sum_p (Nn/p)\ (10^{p-1} + 10^{p-2} + \ldots\ldots + 10 + 1)$ where N is the number of p-partitions of n. As the digits are symetrically placed in any p–partition of n, so this part of the rule is obvious.

We have seen earlier [Note before R. 44c(ii)–45a] that the author of *Prākṛta Paiṅgala* gives a rule for the formation of a figure of numbers called *Mattā Meru* by him. The cells of this figure also contain binomial coefficients. The author of *Prākṛta Paiṅgala* gives[91] a rule for finding the number of variations of *mātrā vṛtta* of n *mātrās* having any particular number of *gurus*, (or *laghus*) in terms of binomial coefficients. Correspondingly, the rule gives a method for finding the number of p–partitions $(0d_i)$, $i = 1, 2$ of n, $p = 1, 2, 3, \ldots\ldots, n$ amongst all partitions $(0d_i)$, $i = 1, 2$ of n as only 1 and 2 enter into any of the partition set.

We have seen that Nārāyaṇa's partition of n for a particular q, is an extension of partition $(0d_i)$, $i = 1, 2$ of n. We have also seen that Nārāyaṇa's sequence of polynomial coefficients for a q is an extenion of the sequence of binomial coefficients. Obviously, then Nārāyaṇa's p-partitions of n, $p = 1, 2, 3, \ldots\ldots, n$ for a particular q should be expressible in terms of the above type of sequence of polynomial coefficients. The rule describes the same. Thus Nārāyaṇa's rule is a generalization of the rule given by the author of *Prākṛta Paiṅgala* for variations of *mātrā vṛttas* to partitions of a number.

Consider the 2nd case of Ex. 34. Here, the greatest digit = 3 and the sum (of the digits, i.e., the given number) = 7. The corresponding *matsya-meru* will be the

same as that obtained in Ex. 13, R. 40–44. Numbers placed in the last i.e., (7 + 1 =) 8th column of the *matsya-meru*, are 6, 16, 15, 6 and 1. So, according to the rule, 6, 16, 15, 6, and 1 are the numbers of p-partitions of 7, for $p = 3, 4, 5, 6$ and 7, in order. Multiplying these (i.e., 6, 16, 15, 6 and 1), by 7 and then dividing the products, separately, by 3, 4, 5, 6 and 7, in order, we get, 14, 28, 21, 7 and 1. Again, multiplying these, separately, by the line of unseparated units for p places (p =3, 4, 5, 6 and 7, in order), i.e., by 111, 1111, 11111, 111111 and 1111111, respectively, we obtain 1554, 31108, 233331, 777777 and 1111111 as the sum of all p-partitions of 7, for $p = 3, 4, 5, 6$ and 7, in order.

Adding these, we get 2154881. So, 2154881 is the sum of all p-partitions of 7 for $q = 3$.

SERIAL ARRANGEMENT FOR PARTITIONS OF A NUMBER

R. 79–80 "First of all write the greatest digit and on its left a digit (or digits) such that the digits, when added, yield the given sum. Starting from the beginning (each time) write the next lesser digit below (the 1st available) digit greater (than 1), remaining digits on the right, as above (and) on the left, digit (or digits, such that) the sum of all the digits equals the given sum. Repeat the process (until) all (the places in an arrangement) are occupied by unities. This famous method has been told by scholars of old dramatic art."

First of all the greatest digit is written. On its left such number of greatest digits and (or) a lesser digit (if required) are written in ascending order so that the sum of all the digits in the arrangement becomes equal to the given number whose partitions are being arranged. Now each time the process starts by replacing the 1st available digit different from 1, by the next lesser digit which is written below. Digits on the right of the replaced digit are written below. Digits on the right of the replaced digit are written without any change.

For writing digits on the left the rule is the same as that meant for the 1st arrangement. The process is, repeated until all the places in an arrangement are occupied by unities only.

The author explains the rule with the help of two examples. In the first case, $q = 7$ and $n = 7$. The arrangement of partitions in the 2nd case is as given below.

Serial Arrangment of Partitions of 7 when $q = 3$, R. 79–80, 2nd Case of Ex. 36–37.

(1) 133	(12) 2122	(23) 2131	(34) 3211
(2) 223	(13) 11122	(24) 11131	(35) 12211
(3) 1123	(14) 1312	(25) 1321	(36) 21211
(4) 313	(15) 2212	(26) 2221	(37) 111211
(5) 1213	(16) 11212	(27) 11221	(38) 13111
(6) 2113	(17) 3112	(28) 3121	(39) 22111
(7) 11113	(18) 12112	(29) 12121	(40) 112111
(8) 232	(19) 21112	(30) 21121	(41) 31111
(9) 1132	(20) 111112	(31) 111121	(42) 121111
(10) 322	(21) 331	(32) 2311	(43) 211111
(11) 1222	(22) 1231	(33) 11311	(44) 1111111

FIGURES OF NUMBERS (CONTINUED) : *UNMERU*

R. 81–83 "(Form horizontal) series of cells, (starting from) 1 (cell) in the beginning (and) increasing by 1 (cell each time, until their number is) 1 more than the sum. Write the *sāmāsikā* sequence in cells of the lowest row, in order and in the others, the arithmetic progression in the reverse order. Naturally, vertically (also) these (numbers) are in arithmetic progression. Omit the cells containing numbers greater than the greatest digit. Thus the *umeru* is formed."

Here, the sum means the number which is to be partitioned. Also, the arithmetic progression is 1, 2, 3, ..., q, (R.10a), the digits greater than q being omitted.

Ex. 38 "O learned, if you know mathematics, tell how the *unmeru* of the two (previous) examples look."

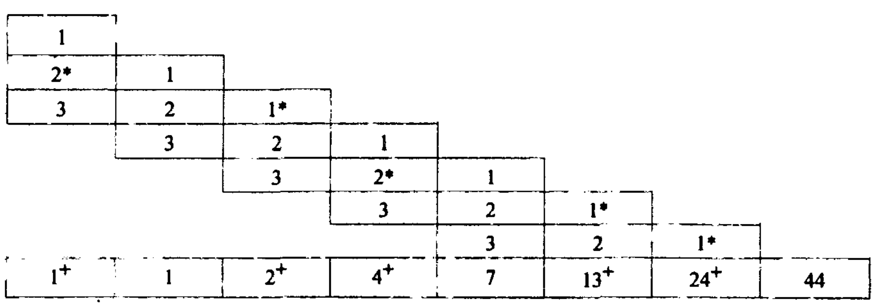


Fig. 8 (R. 81–83, Ex. 38) (*Unmeru* when $n = 7$, $q = 3$)

Here, the reference is to examples 36 and 37, stated earlier. Nārāyaṇa has illustrated the rule by forming the figure of numbers in both the cases. Consider the case when the sum (i.e., the number = n) = 7 and the greatest digit (= q) = 3. For *Unmeru* when $n = 7$, $q = 3$, see Fig. 8.

PARTITION CORRESPONDING TO GIVEN SERIAL NUMBER

R. 84–86(a–b) "Subtract the serial number from the last number of the *sāmāsikā* sequence. Subtract from that (i.e., the remainder), the numbers (of the sequence) nearest to the remainders (one by one) until no subtraction is possible. The digit in the cell common to the row and the column is the 1st digit of the desired partition. In this way, in finding the desired partition, work this process dependent upon *unmeru*, repeatedly (and write) those numbers in a line (from right to left)."

Explanation : First of all corresponding *unmeru* is formed by R. 81–83. Let the serial number be N and the numbers of the *sāmāsikā* sequence be,

$$d_1, d_2, d_3, \ldots, d_r, \ldots d_{n-2}, d_{n-1}, d_n.$$

Suppose, $d_n - N = D$, and, $D - d_{n-1} = D_{n-1}$.

Now suppose that d_{n-2} cannot be subtracted from D_{n-1}. Then the digit common to the column of d_{n-2} and the row just above the row in which the *sāmāsikā* sequence has been written is the last digit of the desired partition.

After that, suppose, $D_{n-1} - d_{n-3} = D_{n-3}$ $\quad D_{n-3} - d_{n-4} = D_{n-4}$ and so on.

But now suppose that d_r cannot be subtracted from the last remainder, so obtained. Then, the digit common to the column of d_r and the row just above 'the uppermost digit' of the column of which a digit of the partition has been obtained earlier is the penultimate digit of the expansion. The process is followed with all the numbers of the *sāmāsikā* sequence and these are subtracted from right to left and digits obtained are also written from right to left.

The author of *Prākṛta Paiṅgala* gives[92] a rule for finding the variation of a *mātrā vṛtta* having n *mātrās* corresponding to a given serial number in the serial arrangement of all variations of the *mātrā vṛtta* without going through the serial arrangement itself. Correspondingly, the rule is meant for finding the partition ($0d_i$), $i = 1, 2$, of n, if the serial numeber of the partition, in the serial arrangement of all such partitions is given. Nārāyaṇa's rule is an extension of the same to partitions of a number. .

Ex. 39 "If you know (how) to free (yourself) from the mathematics of concatenation of numbers, tell separately, how the shape of 36th arrangements (of partitions) in the two previous examples look."

Here, the reference is to Exs. 36–37.

Solution : Let us consider the 2nd case of that example. Here, the number = 7, the greatest digit = 3 and the given serial number = 36 and so the *unmeru* formed by R. 81–83, will be as shown in Fig. 8 in the solution of the 2nd case of Ex. 38. Also, the *sāmāsikā* sequence formed by R.13–15a will be 1, 1, 2, 4, 7, 13, 24 and 44, in this case.

We have, 44 – 36 = 8. Numbers of the *sāmāsikā* sequence, from right to left starting from the penultimate are subtracted from this remainder 8.

Digit obtained from the *unmeru*

24 cannot be subtracted from 8 1

13 cannot be subtracted from 8 1

8 – 7 = 1

4 cannot be subtracted from 1 2

2 cannot be subtracted from 1 1

1 – 1 = 0

1 cannot be subtracted from 0 2

The digits obtained above have been marked by a star mark over them in the *unmeru* (Fig. 8, R. 81–83, Ex. 38). Writing the digits obtained above from right to left we get the desired 36th arrangement of partition as 21211 amidst serial arrangement of all partitions of 7 into digits ≤ 3.

SERIAL NUMBER CORRESPONDING TO GIVEN PARTITION

R. 86c–d "Those numbers (of the *sāmāsikā* sequence) obtained from the *unmeru* due to the digits of the given partition, subtracted from the last number (of the *sāmāsikā* sequence), yield the (desired) serial number."

Explanation : Let there be *r* digits in the given partition. Then *r* corresponding numbers of the *sāmāsikā* sequence are omitted from it in the following manner. Last digit of the partition is taken in the row just above the row in which the *sāmāsikā* sequence has been written. Numbers of the *sāmāsikā* sequence which lies in the same column in which the last digit lies, is left out from the sequence. Next, the penultimate digit of the given partition is taken in the row which is just above the row of 'the uppermost digit' of the column in which the last digit of the given partition lies. The number of the *sāmāsikā*

sequence in the column in which the penultimate digit lies is again left out from the sequence. Thus, moving from right to left the numbers of the *sāmāsikā* sequence corresponding to all digits of the given partition are left out. The remaining numbers of the *sāmāsikā* sequence, excluding the last, are added and the sum is subtracted from the last number of the *sāmāsikā* sequence. The remainder gives the desired number of the given partition amidst serial arrangement of all partitions of the number.

The author of *Prākṛta Paiṅgala*[93] gives a rule for finding the serial number of a given variation in the serial arrangement of all variations of a *mātrā vṛtta* having *n mātrās* without going through the serial arrangement of all variations of the metre. Correspondingly, the rule is for finding the serial number of a given partition ($0d_i$), $i = 1, 2$, in the serial arrangement of all such partitions of n. Nārāyaṇa's rule is an extension of the same to partitions of a number.

Ex. 40 "O expert in mathematics, if you know 'the concatenation of numbers' tell the serial numbers of the (two given partitions) in the two (previous) examples."

Here, the reference is to Exs. 36–37 and to partitions obtained as solutions to Ex. 39.

Let us consider the 2nd case of the examples and the solutions. In this case, the number = 7, the greatest digit = 3 and the solution, i.e., the given partition = 21211. The *unmeru* formed by R. 81–83, will be as shown in Fig. 8 in the solution of the 2nd case of Ex. 38. Also, the *sāmāsikā* sequence formed by R.13–15a will be 1, 1, 2, 4, 7, 13, 24 and 44, in this case.

The numbers of the *sāmāsikā* sequence corresponding to the digits of the given partition, as suggested by the rule and which are to be left out, have been marked by a plus sign over them in the *unmeru*. Sum of the rest numbers, excluding the last, of the *sāmāsikā* sequence (= 1 + 7) = 8. Subtracting 8 from the last number of the sequence, we get, (44 – 8 =) 36 as the desired serial number of the given partition amidst serial arrangement of all partitions of 7 into digits ≤ 3.

PERMUTATIONS OF n (DISSIMILAR) DIGITS TAKING r AT A TIME (CONTINUED)

In the following permutations all digits $\leq q$, take part.

R. 87 "Subtract 1 from the greatest digit in succession. The product of such digits, their number being equal to the number of places, determines the number of permutations. Multiply that by half of 'the greatest digit increased by 1'. (The product) is the sum of digits in a particular place in all permutations. (The sum of all permutations should be obtained by the method stated earlier)."

According to the rule,

(1) $P(q, r) = q(q-1)(q-2) \ldots\ldots (q-r+1)$.

(2) The sum of digits at a particular place in all permutations $= \left(\frac{q+1}{2}\right) . P(q, r)$.

(3) By R. 45–48, the sum of all permutations

$$= \left(\frac{q+1}{2}\right) . (10^{p-1} + (10^{p-2} + 10^{p-3} \ldots\ldots + 10 + 1) . P(q, r) .$$

Rationale : (1) The rule is the well known rule for finding the value of $P(q, r)$.

(2) In all permutations each digit from 1 upto q will be repeated $P(q-1, r)$ times at a particular place. So the sum of digits at a particular place in all permutations

$$= (1 + 2 + 3 \ldots\ldots + q) . P(q-1, r)$$

$$= \frac{q(q+1)}{2} . P(q-1, r) = \left(\frac{q+1}{2}\right) . P(q, r) .$$

Ex. 41 "Tell the number of permutations (etc.) when the greatest digit is 8 and the number of places is 3 (and also), tell quickly, (when) the greatest digit is 9 and the number of places is 6."

COMBINATION OF n DISSIMILAR DIGITS TAKING r AT A TIME (CONTINUED)

In the following combinations all digits from 1 upto q take part. Also, the digits in such combinations are always in ascending order.

R. 88 (a–b) "That (i.e., the number of permutations) divided by the product of number of the arithmetic progression is the number of combinations in ascending order."

Rule for finding the number of permutations has been given in R. 87 and the arithmetic progression has been defined in R. 10a. According to this rule,

$$C(q, r) = \frac{P(q, r)}{1.2 \ldots\ldots r} = \frac{\underline{\lfloor q}}{\underline{\lfloor r} \cdot \underline{\lfloor q-r}}$$

The rule is the well known rule for finding the value of $C(q, r)$.

Ex. 42 "O experienced learned, tell the number of combinations in ascending order in the above stated 2 examples (and also tell), how the serial arrangement (of all combinations) look thereat?"

The reference is to Ex. 41 stated above. Method for the serial arrangement has been given in the next rule [R. 88–89a–89b].

SERIAL ARRANGEMENT OF COMBINATION IN ASCENDING ORDER

R. 88 (c–d)–89(a–b) "(Keep the digits in ascending order). Starting from the beginning (each time), keep the (next) lesser (digit) below the greater; the remaining digits on the right, as above. Write the digits, each less by 1, in the reverse order, on the left. Follow the process of expansion until an arithmetic progression is obtained."

Nārāyaṇa illustrates his rule with the help of both the cases of Ex. 41.

Consider the 1st case of that example. Here, $p = 3$, $q = 8$. The expansion obtained by the rule is as given below.

Solution : Ex. 41, 1st case, expansion when $p = 3$ and $q = 8$.

(1) 678	(15) 158	(29) 257	(43) 146
(2) 578	(16) 348	(30) 157	(44) 236
(3) 478	(17) 248	(31) 347	(45) 136
(4) 378	(18) 148	(32) 247	(46) 126
(5) 278	(19) 238	(33) 147	(47) 345
(6) 178	(20) 138	(34) 237	(48) 245
(7) 568	(21) 128	(35) 137	(49) 145
(8) 468	(22) 567	(36) 127	(50) 235
(9) 368	(23) 467	(37) 456	(51) 135
(10) 268	(24) 367	(38) 356	(52) 125
(11) 168	(25) 267	(39) 256	(53) 234
(12) 458	(26) 167	(40) 156	(54) 134
(13) 358	(27) 457	(41) 346	(55) 124
(14) 258	(28) 357	(42) 246	(56) 123

MOVEMENT OF SHOTS FOR FINDING SERIAL NUMBER OF A GIVEN COMBINATION AND VICE-VERSA

R. 89 (c–d)–90 "Form cells in rows such that their number in each row is equal to 'the greatest digit'. Write the arithmetic progression with 1 as the 1st term and the same as the common difference in cells of the 1st row. Starting from the 1st, keep shots (equal to the greatest digit) in cells of the 1st row. Move the shots continuously starting from the 1st and then that in front of it."

Explanation : Since shots are kept in cells from the beginning, initially the last shot is kept ahead and then the penultimate shot is moved. After that the shot before the penultimate shot is moved and so on and so forth. After moving the 1st shot, in the next step the last shot is moved ahead and other shots are brought to their initial positions. Next, the penultimate shot is moved and the process is repeated until the shots occupy continuous cells. Again in the next step, the last shot is moved ahead and other shots are brought to their 'initial positions'. Thus, the process is repeated again and again.

It can be easily seen that the last short remains under the digit $(p + r)$ for $C(p + r - 1, r)$ movements barring the initial position. For any such position of the last shot the penultimate shot remains under the digit $(p + r - 1)$ for $C(p + r - 2, r)$ movements barring the initial position. Similar rule holds for positions of shots prior to these two shots.

COMBINATION CORRESPONDING TO A GIVEN SERIAL NUMBER

R. 91 "Digits above the shots, after such number of their movements as are equal to that (i.e., the given serial number) subtracted from the total number of combinations, definitely happen to be the desired combination."

Ex. 43–44(a–b) "O elder master of expansion, in those 2 (previous) examples, tell here quickly, what are the structures of the 40th combinations."

Here, the example referred to is Ex. 41 stated earlier. Consider the 1st case. Here, $p = 3$ and $q = 8$ and so the arithmetic progression is q, 2, 3, 4, 5, 6, 7 and 8 (R. 10a). Also total number of combinations obtained by the rule (as stated earlier) is 56. Subtracting the given serial number from the total number of combinations, we get, 16. Three shots are kept below 1, 2, 3 of the arithmetic progression and the shots are moved as described above. In the initial position, since the shots are kept in continuation from the beginning, in practice, the process has to be started by moving the last shot (instead of the shot lying in the beginning as directed by the rule) and then the shot towards the left and so on. Nevertheless, the rule is that (whenever possible) the 1st shot from the beginning is to be moved. When such a movement is not possible then and then only, the 1st shot towards its right side, whose movement is possible, is to be moved and so on.

Different stages in the movement of shots will be clear from the figure given below. Digits above the shots, after 16 movements give the desired combination as 156.

Laḍḍuka Paṅkti	1	2	3	4	5	6	7	8
Initial Position	0	0	0					
Shots after 1st Movement	0	0		0				
” ” 2nd ”	0		0	0				
” ” 3rd ”		0	0	0				
“ ” 4th “	0	0			0			
” “ 5th ”	0		0		0			
“ ” 6th “		0	0		0			
” “ 7th ”	0			0	0			
“ ” 8th “		0		0	0			
” “ 9th ”			0	0	0			
“ ” 10th “	0	0				0		
” “ 11th ”	0		0			0		
“ ” 12th “		0	0			0		
” “ 13th ”	0			0		0		
“ ” 14th “		0		0		0		
” “ 15th ”			0	0		0		
“ ” 16th “	0				0	0		

Fig. 9 Movement of shots when $p = 3$ and $q = 8$. R. 89–92, Exs. 43–44 (a–b) and 44 (c–d)

SERIAL NUMBER CORRESPONDING TO A GIVEN COMBINATION

R. 92 "Subtract the number of movements (of shots below) the digits of the given combination (in bringing them into cells in the beginning) from that (i.e., from the total number of combinations). The remainder added to 1 is the measure of the serial number of the given combination."

Here, the initial position has to be counted as the shots after the 1st movement and so on.

A remarkable point which is to be noted here is the relation of the above two rule (R. 91 and R. 92) for finding the structure of a combination whose serial number is given and vice-versa with Nārāyaṇa's 'rth sum of a natural number p' which is equal to $C(p + r, r + 1)$. This is so, because, as has been observed earlier, the last shot remains under the digit $(p + r)$ for $C(p + r - 1, r)$ movements of shots barring the initial position and $C(p + r - 1, r)$ is the $(r - 1)$th sum of p. Similarly, for any position of the last shot, the penultimate shot remains under the digit $(p + r - 1)$ for

$C(p + r - 2, r)$ movements of shots, barring the initial position and $C(p + r - 2, r)$ is the $(r - 1)$th sum of $(p - 1)$ and so on and so forth.

Ex. 44 (c–d) "(If) the structure of those 2 combinations are known, tell quickly, their serial numbers."

Here, combinations referred to are the combinations obtained as solutions of Ex. 43–44(a–b). Consider the 1st combination which is 156. The combination will be obtained after 17 movements of the shots over the arithmetic progression 123. So, the desired serial number (= 56 – 17 + 1) = 40.

CASES OF FAILURES OF RULES

R. 93 "If the greatest digit be more than 9, the places of digits are deformed. For that reason, only the number of permutations is obtained correctly by writing the digits upto the greatest, separately."

Ex. 45 "O learned, if the numbers be 9 and more and the number of places be 3, tell here, the number of permutations and show how the total sum (etc.) look."

Here, the number of permutations is 6 (R. 45b–48), which is correct if the numbers 9, 10 and 11 (say), are written, separately. However, since the places of digits are deformed, the sum of digits at a particular place etc. are not obtained, correctly."

CONCLUDING REMARKS

R. 94–97 "There are many kinds of *prastāras* (i.e. expansions), *merus* (i.e., figures of numbers) and *patākās* (i.e., another kind of figures of numbers), (whose) measures are the boundless ocean of mathematics. Here, 'the concatenation of numbers' in mathematics has been told with the help of 1 etc. digits upto 9 as the greatest digit in brief. (I) should be excused for saying – as is the world created by God without the Creator (i.e., nothing, so is mathematics) without digits less than 10 (i.e., nothing).

Here, in the chapter called 'the concatenation of numbers' in mathematics, there is neither multiplication nor division, (nor square) nor square-root nor cube[94] nor cube-root nor unit fractions nor chapter on fractions. (Moreover, here there is) neither rule of three nor rule of five, nor (rules for finding the area, the volume etc. nor operations (for finding) the unknown by equalization.

This chapter on 'the concatenation of numbers' has been written with rules in detail for tying manifold (and) strongly, bad mathematicians, who are wicked, mischievous, stupid (and) are[95] blind with false sense of pride like an elephant with ichor."

REFERENCES

1. P. Singh's translation of the *Gaṇita Kaumudī* Chapters I, II and III was published in *Gaṇita Bhāratī* Vol. 20 (1998), pp. 25–82, chap. IV in Vol. 21 (1999), pp. 10–73 and chapters V to XII, in Vol. 22 (2000), pp. 19–85.

2(a). Divivedi, Rajendra (Translator), *Bhārtīya Vijyāna Ke Karṇdhāra* (Hindi), Research Institute of Ancient Scientific Studies, New Delhi, 1967, pp. 157–159 and 183–187.

2(b). Biggs, N.L., The Roots of Combinatorics, *Historia Mathematica*, Vol. 6 (1979), pp. 109–136.

3. For details, ref. Vinaya Sagar, M., *Vṛtta Mauktika*, Rajasthan Oriental Research Institute, Jodhpur, 1965, 531.

4. Ref. 3, *Bhūmikā*, pp. 9–11.

5. *BSS*, *Chandaścityuttarādhyāya*, pp. 357–358.

6. *GSS*, pp. 146–147 and 177–180.

7. *GK*, chap. XIII.

8–10. *Rm*, 38–40.

11. *BSS*, 358.

12–14. *Pr. Pa*, 41–43.

15(a). *Pr. Pa*, 41–43.

15(b). For details, see Singh, Parmanand, The Socalled Fibonacci Numbers in Ancient and Medieval India, *Historia Mathematica*, Vol. 12 (1985), 229–244.

16. *BSS*, 357–358.

17. *Cf. L*, 166–67 and 170.

18–19. See also, *L*, 166–67.

20–24. Ref. 2(b), 109–136.

25. *Cf. L*, 168.

26. See also, *L*, 168.

27. Ref. 2(a) 183–187.

28–29. Ref. 2(b) 109–136.

30. *Cf. GSS*, 146–147; *PG*, 58–59; *MS*, chap. XV, R .45b–46a; *L*, 61–65.

31. Jha, Sita Ram (Ed.) *Bṛhajjātaka of Varāhamihira with the commentary of Bhaṭṭotapala*, Thakur Prasad and Sons, Varanasi, 1974, 239–40 (R. 19ff).

32. *Cf. GSS*, 147; *GP*, 58.

33. *P(Y)*, 103–105 (R. 2–5); *P(H)*, 190–191 (R. 20–23).

34. *Bh.*, 1192–1200 (R. 113b–115); *Jd*, 37(R .2–3); *Jk*. 69(R .2–4) and *Vr*, 92 (R .1–2).

35. *Ag*, 487 (R. 1); *VP*, chap. 3 (R. 14–16) and *NP*, 389–90 (R. 14b–15).

36. *Jn*, 97–106 (R. 2–10); *VJS*, 86 (R. 13) and *He*, 113 (R .2).

37. *BSS*, 357–58.

38. *GSS*, 177–78 (R. $333\frac{1}{2}$–$334\frac{1}{2}$).

39(a). *P(Y)*, 111–12 (R. 16).

39(b). *P(H)*, 195 (R. 35)

40. *P(Y)*, 112.

41. *P(H)*, 195.

42. *Ag*, 487 (R. 3–4).

43(a). *VJS*, 84–85 (R. 10–11). Figure by Gopāla, 85.

43(b). Sharma, Aryendra and others, *Vṛtta Ratnākara of Kedāra* with the commentary *Tātparya* of Trivikrama, Sanskrit Academy, Osmania University, Hyderabad–7, 1969, vi, 319–322 (R. 239–40 ff); Figure by Trivikrama, 321.

43(c). *Pr. Pa.* 37–39 (R. 44).

43(d). For details, refer, Singh, Parmanand, Binomial and Multinomial Theorems in India and Central Asia, *Interaction between Indian and Central Asian Science and Technology in Medieval Times* (Indo Soviet Joint Monograph Series) Vol. I, 297–298.

44. *Bh.*, 1192–1200 (R. 122).

45. *BSS*, 358 (R. 15).

46. *Jn*, 97–106 (R. 18); *VJS*, 83–84 (R. 7–9); *Jd*, 39–40 (R .8–10); *Rm*, 37 (R. 16–17); *Jk*, 70 (R. 9–10); *Vr*, 92 (R. 5–6); Ref.43(b), 320–321 (R. 239–240 ff); *He* 113 (R. 7–8).

47(a) Tripathi, A.B., *Bṛhat Saṁhitā* of Varāhamihira with the commentary of Bhaṭṭotpala, Varanaseya Sanskrit Vishvavidyalaya, Varanasi, Part II 1968, 844–46.

47(b). Also ref. 31, 239–40 (R. 19ff.).

47(c). For details ref. 43d, 300–302.

48(a). *Rm*, 39 (R. 11) and 40.

48(b). For details, ref. 43, 299.

49. Apte, V.G.(Ed.), *Līlāvatī* of Bhāskara II (with the commentaries *Budhivilāsinī* of Gaṇeśa and *Līlāvatī vivaraṇa* of Mahīdhara) Anandasrama Sanskrit Granthavali No. 107, Poona, Part I, 107 (R. 114).

50(a). Ref. 47a, 844–46 [R. 22(c–d)].

50(b). See also, ref. 31, 239–40 (R. 19 ff).

50(c). For details, ref. 43(d), 300–302.

51. For details, ref. 43(d), 300–302.

52(a). *P(Y)*, 107–111 (R. 10–14).

52(b). See also, *HHM* (I), 76–77.

53. *Bh*, 1192–1200 (R. 122–23); *Jd*, 40(R .11); *Jk*, 70(R .10b); *Vr*, 92 (R .7).

54(a). Yadegiri, M., The Binomial Theorem: A Widespread Concept in Medieval Islamic Mathematics; *Historia Mathematica*, 7 (1980), 401–406.

54(b). Ref. 2(b), 109–136.

55. Ref. 54(a), 401–406.

56(a). Ref. 2(b), 109–136.

56(b). Ref. also, 54(a), 402.

57. Ref. 54(a), 402–404.

58(a). Ref. 2(b), 109–136.

58(b). Ref. 54(a), 402.

59. Ref.54(a), 402.

60. Ref. 2(b), 109–136.

61. Needham, Joseph, *Science and Civilization in China*, Vol. III Mathematics and the Science of the Heavens and the Earth, Cambridge University Press, Cambridge, 1959, 137.

62. Ref. 61, 134.

63(a). Smith, D.E., *History of Mathematics*, Dover Publications, New York, Vol. 2, 1958, 508.

63(b). Ref. 61, 134.

64. Ref. 63(a), 509.

65(a). Eves Howard, *An Introuction to the History of Mathematics*, Holt, Rinehart and Winston, New York, 1976, 4th edition, 217.

65(b). Ref. 63(a), 510.

66–67. Ref. 63(a), 510.

68. Ref. 65(a), 263.

69–70. Ref. 63(a), 511.

71. Ref. 8, 38–40.

72. *P(Y)*, 105–106 (R .6–7); *Bh*, 1192–1200 (R. 128); *Ag*, 487 (R .2a); *Jn*, 97–106 (R .11–13); *VJS*, 92–94 (R. 30–34); *Jd*, 38 (R. 4–5); *Rm*, 37 (R. 12–13); *Jk*, 69 (R. 7); *Vr*, 92 (R. 3).

73. *BSS*, 357–58 (R. 8 and R .13).

74. *GSS*, 178 (R. 334 $\frac{1}{2}$).

75. *P(Y)*, 106–107 (R. 8–9); *Ag*; 487 (R .2–3); *Jn*, 97–106 (R .14–16); *VJS*, 92 (R .35); *Rm*, 37 (R. 14–15).

76. *BSS*, 357–58 (R. 14).

77. *GSS*, 179 (R. 335 $\frac{1}{2}$); *Jd*, 38 (R .6–7); *Jk*, 70 (R .8); *Vr*, 92 (R .4).

78(a). Ref. 2(b), 109–136.

78(b). See also, Gupta, R.C., The Last Combinatorial Problem in Bhāskara's *Līlāvatī*, *GB*, Vol. 18, Nos.1–4 (1996), 15.

79(a). *P(Y)*, 105.

79(b). For details, ref. 15(b).

80(a). *P(Y)*, 41; *P(H)*, 46–47.

80(b). For details, ref. 15(b).

81. *Bh*, 1192–1200 (R. 113).

82. *VP*, chap. 3 (R. 17)

83. *VJS*, 101 (R. 49).

84(a). *VJS*, 101 (R. 49 ff.).

84(b). For details, ref. 15(b).

85(a). Velankar, H.D. (Ed.), *Chandonuśasana* (Sanskrit) of Hemacandra Sūri, Bhartiya Vidya Bhavan, Bombay–7, 1961, 239.

85(b). For details ref. 15(b).

86. ————, *Kavidarpaṇa* (Prakrit) of an unknown author with an ancient commentary, Rajasthan Oriental Research Institute, Jodhpur, 1962, chap. VI, 67–68 (R. 3).

87. *Pr. Pa.* 31–47.

88. See, *L*, 170–72 (R. 274), for an alternative rule for number of p–partitions of n.

89. *Pr. Pa.* 44–45 (R .49).

90(a). For an alternative rule for p–partition of n, see, *L*, 170–72 (R. 274).

90(b). Also, ref. 78(b), 15–19.

91. *Pr. Pa.*, 41–43 (R .46–47).

92. *Pr. Pa.*, 32–33 (R. 40–41).

93. *Pr. Pa.*, 31 (R. 39).

94–95. *Cf. L*, 172 (R .276).

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INTERNATIONAL CONFERENCE OF I.S.H.M.

An International Conference on the History and Heritage of Mathematical Sciences is being organised by the *Indian Society for History of Mathematics* and the *Sukrtindra Oriental Research Institute* at Cochin in collaboration with other Institutions and Organisations. The Conference will be held at *Cochin* under the leadership of Dr. T. Thrivikraman (Cochin University), Dr. V. N. Bhat (Sukrtindra Oriental Research Institute) and Dr. V. Madhukar Mallayya (Mar Ivanios College) from 19 to 22 December 2002 and will cover all aspects of the History and Heritage of Mathematical Sciences. The role of Technical Literature in Sanskrit and other languages and the relevance of Traditional Mathematics in Modern Science and Technology will be highlighted. The official language of the Conference will be English. For further information contact Dr. V. Madhukar Mallayya at the following :

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